On feedback target control for uncertain discrete-time systems through polyhedral techniques

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Problems of feedback target control for linear and bilinear discrete-time systems under uncertainties and state constraints are considered. We continue the development of methods of control synthesis using polyhedral (parallelotope-valued) solvability tubes. The paper deals with two types of problems, where the controls appear either additively or in the system matrix (i. e., in the coefficients of the system). Both problems are considered for systems with parallelotope-bounded additive uncertainty and with interval uncertainties in the coefficients. Moreover the systems are considered under constraints on the state. The techniques for calculation of the polyhedral solvability tubes by the recurrent relations are presented. Control strategies, which can be constructed on the base of the mentioned polyhedral tubes by explicit formulas, are proposed. Illustrative examples are given.

Keywords: discrete-time systems, uncertain systems, state constraints, control synthesis, solvability tubes, parallelotopes, parallelepipeds, interval analysis.

Introduction

We consider problems of feedback target control for linear and bilinear discrete-time systems with a fixed terminal time under uncertainties and geometric (hard) time-dependent constraints on the states. There exist a number of approaches to the solution for this kind of problems, including those for differential systems, based on construction of solvability tubes [1 − 3]. The problem statement for differential systems, approaches to their solution and the tight interconnections between solvability tubes (in other terms, maximal stable bridges, Krasovskii's bridges, backward reachable tubes), the Pontryagin alternated integral, Hamilton-Jacobi-Bellman equations, and funnel equations can be found in [2 − 5]. A problem of target control, which is close in some sense, was considered in viability theory for time-invariant systems with a given target set under a time-independent state constraint, where a set of initial states (so called capture basin) should be constructed in a state space [6, 7]. Since practical construction of the trajectory tubes for different problems in control theory (in particular, the solvability tubes, reachable tubes, viable trajectory tubes) as well as of so called "kernels" from viability theory may be cumbersome, various numerical methods have been developed. In particular, these are methods for approximating the set-valued solutions and for numerical solution of the above mentioned equations, including methods based on approximations of sets either by arbitrary polytopes with a large number of vertices or by unions of a large number of points [7 − 14] (here and below, we note, as examples, only some references from numerous publications; see also references therein). Such methods are designed to obtain the most accurate approximations of the solution sets. But they can require a lot of calculations, especially for large dimensional systems.

A number of methods are based on using estimates of the sets by domains with a fixed shape, such as ellipsoids and parallelotopes [2, 3, 5, 8, 15–26]. The main advantage of such techniques is that they enable us to get approximate/particular solutions by rather simple means (up to explicit formulas). More accurate approximations may be obtained by using families (varieties) of such simple estimates (as suggested by A.B. Kurzhanski) [2,3,5,18–26].

The techniques based on using ellipsoids and parallelotopes are ideologically close to methods of interval analysis [27, 28]. Some applications of the methods of interval analysis to solving control problems can be found in [29–33]. Some of the proposed methods can give rather conservative estimates for the sets under consideration due to the well-known "wrapping effect" [27, 28]. Several techniques have been proposed to reduce this effect, in particular, methods based on a partitioning process and using so called subpavings (sets of non-overlapping boxes or interval vectors) [31, 33]. Such methods can give rather accurate guaranteed inner and outer approximations but may require extensive computation and memory for large dimensional systems.

As for solving the feedback target control problems for differential and discrete-time systems, constructive computation schemes that employ ellipsoidal techniques were proposed [2,3,5,18,19] and then expanded to the polyhedral techniques [20,22,24–26] (this had required the development of a quite different techniques).

In the present paper, we continue the development of methods of the polyhedral control synthesis for linear and bilinear discrete-time uncertain systems using polyhedral (parallelotope-valued) solvability tubes. The paper deals with two types of problems, where the controls appear either additively or in the system matrix (i. e., in the coefficients of the system). Both problems are considered for systems with parallelotope-bounded additive uncertainty and with interval uncertainties in the coefficients (the matrix uncertainty). Moreover the systems are considered under time-dependent geometric (hard) constraints on the state, where the state constraints are described in terms of zones (i. e., intersections of strips). Note that the systems with controls (or uncertainties) in the system matrix are of bilinear type and have properties of nonlinear systems. In particular, it is known that reachable sets of systems with uncertain matrices can be non-convex in contrast to reachable sets of linear systems with convex constraints on controls and initial states (see, for example, [10, 34] and reference [5] from [15]). The same is true for the solvability tubes even for the case without uncertainties and state constraints because such solvability tubes are backward reachability tubes. The key issue here is to find suitable techniques which can produce solutions to the problems without being too computationally demanding. Recall that in [22] only the first of the mentioned two problems was considered for the case without the matrix uncertainty, while in [24–26] these problems were considered without state constraints. In the present paper, the techniques for calculation of the polyhedral solvability tubes by the recurrent relations are presented. Control strategies, which can be constructed on the base of the mentioned polyhedral tubes, are proposed. In contrast to [20, 22], these control strategies can be calculated by explicit formulas. Illustrative examples are given.

We use the following notation: $\mathbb{R}^n$ is the n-dimensional vector space; $(x, y) = x^\top y$ is the scalar product for $x, y \in \mathbb{R}^n$; $\top$ is the transposition symbol; $\|x\|_2 = (x^\top x)^{1/2}$,

$\|x\|_\infty = \max\limits_{1\le i\le n} |x_i|$ are the vector norms for $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$; $\mathrm{e}^i = (0, \dots, 0, 1, 0, \dots, 0)^\top$ is the unit vector oriented along the axis x_i (the unit stands at position i); $\mathrm{e} = (1, 1, \dots, 1)^\top$; $\mathbb{R}^{n\times m}$ is the space of real $n\times m$-matrices $A = \{a_i^j\} = \{a^j\}$ with elements a_i^j and columns a^j (the upper index numbers the columns and the lower index numbers the components of vectors); I is the identity matrix; 0 is the zero matrix (vector); $\mathrm{Abs}\, A = \{|a_i^j|\}$ for $A = \{a_i^j\} \in \mathbb{R}^{n\times m}$; $\mathrm{diag}\, \pi$, $\mathrm{diag}\, \{\pi_i\}$ are the diagonal matrix A with $a_i^i = \pi_i$ (π_i are the components of the vector π); $\det A$ is the determinant of $A \in \mathbb{R}^{n\times n}$; $\|A\| = \max\limits_{1\le i\le n} \sum\limits_{j=1}^{m} |a_i^j|$ is the matrix norm for $A\in\mathbb{R}^{n\times m}$ induced by the vector norm $\|x\|_\infty$; and the notation $k = 1, \dots, N$ is used instead of $k = 1, 2, \dots, N$ for brevity.

1. Problems formulation

Let $x \in \mathbb{R}^n$ denote the state. Consider the system

$$\begin{gathered} x[k] = (A[k] + V[k] + U[k])\, x[k-1] + B[k]u[k] + C[k]v[k], \quad k = 1, \dots, N, \\ x[N] \in \mathcal{M} \end{gathered} \tag{1}$$

with a given terminal (target) set $\mathcal{M}$. Here $A[k] \in \mathbb{R}^{n\times n}$, $B[k] \in \mathbb{R}^{n\times n_u}$, $C[k] \in \mathbb{R}^{n\times n_v}$ are given matrices; $U[k] \in \mathbb{R}^{n\times n}$ and $u[k] \in \mathbb{R}^{n_u}$ serve as controls and satisfy either (2) or (3):

$$U[k] \equiv 0, \quad u[k] \in \mathcal{R}[k] \subset \mathbb{R}^{n_u}, \quad k = 1, \dots, N, \tag{2}$$

$$U[k] \in \mathcal{U}[k] = \{U \in \mathbb{R}^{n\times n} |\, \mathrm{Abs}\, (U - \tilde{U}[k]) \le \hat{U}[k]\}, \quad u[k] \equiv 0; \tag{3}$$

$v[k] \in \mathbb{R}^{n_v}$ (unknown but bounded disturbances) and $V[k] \in \mathbb{R}^{n\times n}$ (matrix uncertainties) are subjected to given set-valued constraints:

$$v[k] \in \mathcal{Q}[k] \subset \mathbb{R}^{n_v}, \quad k = 1, \dots, N, \tag{4}$$

$$V[k] \in \mathcal{V}[k] = \{V \in \mathbb{R}^{n\times n} |\, \mathrm{Abs}\, (V - \tilde{V}[k]) \le \hat{V}[k]\}, \quad k = 1, \dots, N. \tag{5}$$

The functions $v[\cdot]$ and $V[\cdot]$ satisfying (4) and (5) are called *admissible*. Matrix and vector inequalities ($\le, <, \ge, >$) here and below are understood component-wise.

Below we consider the following cases: (A) *without uncertainty:* when v and $V \equiv 0$ are given functions, i. e., $\bar{Q} \equiv 0$, $\tilde{V} \equiv \hat{V} \equiv 0$; (B) *under uncertainty* including the following two subcases: (B,i) *only additive uncertainty* ($V \equiv 0$); (B,ii) *also matrix uncertainty* ($V \not\equiv 0$).

The system is complicated by the state constraints:

$$x[k] \in \mathcal{Y}[k] \subset \mathbb{R}^n, \quad k = 0, \dots, N-1. \tag{6}$$

We presume the given sets $\mathcal{R}[k]$ to be parallelepipeds, $\mathcal{Q}[k]$ to be parallelotopes, $\mathcal{M}$ to be a parallelepiped, and $\mathcal{Y}[k]$ to be zones. Let us recall the definitions of the objects we have just mentioned and will use below.

By a *parallelepiped* $\mathcal{P}(p, P, \pi) \subset \mathbb{R}^n$ we mean a set such that $\mathcal{P} = \mathcal{P}(p, P, \pi) = \{x \in \mathbb{R}^n |\, x = p + \sum\limits_{i=1}^{n} p^i \pi_i \xi_i,\ \|\xi\|_\infty \le 1\}$, where $p \in \mathbb{R}^n$; $P = \{p^i\} \in \mathbb{R}^{n\times n}$ is a nonsingular matrix ($\det P \ne 0$) such that $\|p^i\|_2 = 1$; $\pi \in \mathbb{R}^n$, $\pi \ge 0$; the condition $\|p^i\|_2 = 1$ may be omitted to simplify formulas. It may be said that p determines the center of the parallelepiped, P is the orientation matrix, p^i are the "directions", and π_i are the values of its "semi-axes". We call a parallelepiped *nondegenerate* if all $\pi_i > 0$.

By a *parallelotope* $\mathcal{P}[p, \bar{P}] \subset \mathbb{R}^n$ we mean a set $\mathcal{P} = \mathcal{P}[p, \bar{P}] = \{x \mid x = p + \bar{P}\xi, \|\xi\|_\infty \le 1\}$, where $p \in \mathbb{R}^n$ and $\bar{P} = \{\bar{p}^i\} \in \mathbb{R}^{n\times m}$, $m \le n$. We call a parallelotope $\mathcal{P}$ *nondegenerate* if $m = n$ and $\det \bar{P} \ne 0$.

By a *zone* (or *m-zone*) $\mathcal{S} = \mathcal{S}(c, S, \sigma, m) \subset \mathbb{R}^n$ we mean an intersection of $m \le n$ *strips* Σ^i: $\mathcal{S} = \mathcal{S}(c, S, \sigma, m) = \bigcap_{i=1}^{m} \Sigma^i$, $\Sigma^i = \Sigma(c_j, s^j, \sigma_j) = \{x \mid |(x, s^i) - c_i| \le \sigma_i\}$, where $c \in \mathbb{R}^m$; $S = \{s^i\} \in \mathbb{R}^{n\times m}$, vectors s^i are linearly independent; $\sigma \in \mathbb{R}^m$, $\sigma \ge 0$.

Each parallelepiped $\mathcal{P}(p, P, \pi)$ is a parallelotope $\mathcal{P}[p, \bar{P}]$ with $\bar{P} = P \cdot \operatorname{diag} \pi$. Each nondegenerate parallelotope is a parallelepiped with $P = \bar{P} \operatorname{diag}\{\|\bar{p}^i\|_2^{-1}\}$, $\pi_i = \|\bar{p}^i\|_2$ or, in a different way, with $P = \bar{P}$, $\pi = \mathrm{e}$, where $\mathrm{e} = (1, \dots, 1)^\top$. Each parallelepiped is a zone, and vice versa if $m = n$ (corresponding formulas can be found in [21, p. 65]).

Let us consider the following problem (with additive controls u) similar to [2, 3, 19].

Problem 1. Let $U[k] \equiv 0$. For any i, $0 \le i \le N-1$, find a solvability set $\mathcal{W}[i]$ and a feedback control strategy $u = u[k, x]$ with $u[k, x] \in \mathcal{R}[k]$ such that each solution $x[\cdot]$ to

$$x[k] = (A[k] + V[k])x[k-1] + B[k]u[k, x[k-1]] + C[k]v[k], \quad k = i+1, \dots, N,$$

that start from any $x[i] \in \mathcal{W}[i]$ would reach the target set ($x[N] \in \mathcal{M}$) and satisfy state constraints (6) whatever are admissible functions $v[\cdot]$, $V[\cdot]$ subjected to (4), (5).

Similarly to [2,3], we say that the multivalued function $\mathcal{W}[k]$, $k = 0, \dots, N$, is a *solvability tube* $\mathcal{W}[\cdot]$[1].

The tube $\mathcal{W}[\cdot]$ describes the set of all those positions from which the system can reach the given target set in specified time, ensuring viability under uncertainties (contractions) and using all possible controls.

Everywhere below we accept the following assumption.

Assumption 1. The set $\mathcal{M} = \mathcal{P}(p_{\mathrm{f}}, P_{\mathrm{f}}, \pi_{\mathrm{f}}) = \mathcal{P}[p_{\mathrm{f}}, \bar{P}_{\mathrm{f}}]$ is a nondegenerate parallelepiped ($\det \bar{P}_{\mathrm{f}} \ne 0$); the sets $\mathcal{R}[k] = \mathcal{P}[r[k], \bar{\mathcal{R}}[k]]$ are parallelepipeds; $\mathcal{Q}[k] = \mathcal{P}[q[k], \bar{Q}[k]]$ are parallelotopes; $\mathcal{Y}[k]$ are zones: $\mathcal{Y}[k] = \bigcap_{i=1}^{m} \Sigma^i[k]$, $\Sigma^i[k] = \Sigma(c_j[k], s^j[k], \sigma_j[k]) = \{x \mid |(x, s^i[k]) - c_i[k]| \le \sigma_i[k]\}$ (or $\mathcal{Y}[k] = \mathbb{R}^n$); all matrices $D[k] = A[k] + \tilde{V}[k] + \tilde{U}[k]$ are nonsingular[2].

The solution to Problem 1 for cases (A), (B,i) (i. e., without matrix uncertainty) is known (see [19] and [22] for the cases without and under disturbances $v[\cdot]$ respectively) and may be

[1] Recall that there are also known other terms for constructions of this kind, for example, maximal stable bridges [1, 12], Krasovskii bridges [2], solvability tubes under counteraction [2], tubes of strongly invariant sets under counteraction [2, p. 72], backward reachability (or reach) tubes [3], attainability tubes in backward time [2] (or, for the case under state constraints, also viability tubes and viable trajectory tubes [3]). Along with the term solvability sets [2, 3, 11, 19] for the cross-sections of the mentioned tubes the following terms are also used: backward reachable (or reach) sets [3, 13] and, for the case without uncertainties and state constraints, also weakly invariant sets [3], and so on. The reach sets are also known as the attainability sets/domains [2, 3, 9].

[2] Note that the nonsingularity of matrices $D[k]$ is not too restrictive for many important cases. In particular, it holds for the case when systems (1)–(5) are some approximations of similar differential systems. For example, for the Euler discretization with the step h_N (see Remark 3.3 below and [20]), matrices $D[k] = A[k] + \tilde{V}[k] + \tilde{U}[k]$ are of the type $D[k] = I + h_N(A(t_{k-1}) + \tilde{V}(t_{k-1}) + \tilde{U}(t_{k-1}))$, where I is the identity matrix. Therefore such matrices $D[k]$ are nonsingular if the discretization step h_N is sufficiently small.

described by the following relations:

$$\begin{gathered}\mathcal{W}[k-1] = A[k]^{-1}((\mathcal{W}[k]\dot{-}C[k]\mathcal{Q}[k])-B[k]\mathcal{R}[k])\cap\mathcal{Y}[k-1], \quad k = N,\ldots,1, \\ \mathcal{W}[N] = \mathcal{M},\end{gathered} \tag{7}$$

$$u[k,x] \in \mathcal{U}[k,x] = \mathcal{R}[k] \cap \{u \mid B[k]u \in (\mathcal{W}[k]\dot{-}C[k]\mathcal{Q}[k]) - A[k]\,x\}, \tag{8}$$

where $u[k,x]$ is any function with values in $\mathcal{U}[k,x]$.

We have the recurrent relations for $\mathcal{W}[\cdot]$, which involve operations with sets such as *Minkowski's sum* ($\mathcal{X}^1+\mathcal{X}^2 = \{y \mid y = x^1+x^2,\ x^k \in \mathcal{X}^k\}$), *Minkowski's difference* ($\mathcal{X}^1\dot{-}\mathcal{X}^2 = \{y \mid y+\mathcal{X}^1 \subseteq \mathcal{X}^2\}$), and *intersection of sets.* The sets $\mathcal{W}[k]$ are not parallelotopes in general, and exact construction of $\mathcal{W}[\cdot]$ by these formulas can be very cumbersome. Even more difficulties arise for the cases with uncertainties/controls in matrices.

Therefore, in the present paper, following ideas from [2, 3], the techniques of control synthesis are elaborated, which use internal (inner) estimates for the solvability tubes.

We call $\mathcal{P}^-$ ($\mathcal{P}^+$) an *internal (external) estimate for* $\mathcal{Q} \subset \mathbb{R}^n$ if $\mathcal{P}^- \subseteq \mathcal{Q}$ ($\mathcal{P}^+ \supseteq \mathcal{Q}$)[3].

In [22], the families of external $\mathcal{P}^+[\cdot]$ and internal $\mathcal{P}^-[\cdot]$ parallelepiped-valued and parallelotope-valued (shorter, *polyhedral*) estimates for $\mathcal{W}[\cdot]$ from (7) (for the case without matrix uncertainty) were introduced ($\mathcal{P}^-[k] \subseteq \mathcal{W}[k] \subseteq \mathcal{P}^+[k]$, $k = 1,\ldots,N$), and control strategies $u[k,x]$ were proposed, which may be constructed by solving systems of linear inequalities. Note that if the initial point $x[0] = x_0$ is out of at least one of the external estimates $\mathcal{P}^+[0]$, then there is no guarantee that it can be steered to the terminal set for any disturbances, and if it belongs to one of the internal estimates $\mathcal{P}^-[0]$, then it can reach $\mathcal{M}$ using the mentioned control strategy.

Now let us consider two problems, which concerns all above cases (A) – (B,ii), where the controls appear either additively or in the system matrix.

Problem 2. Let $U[k] \equiv 0$. Find a polyhedral tube $\mathcal{P}^-[\cdot]$ that satisfies $\mathcal{P}^-[N] = \mathcal{M}$ and $\mathcal{P}^-[k] \subseteq \mathcal{Y}[k]$, $k = 0,\ldots,N-1$, and find a corresponding feedback control strategy $u = u[k,x]$ such that $u[k,x] \in \mathcal{R}[k]$ for $x \in \mathcal{P}^-[k-1]$, $k = 1,\ldots,N$, and each solution $x[\cdot]$ to

$$x[k] = (A[k] + V[k])\,x[k-1] + B[k]u[k,x[k-1]] + C[k]v[k], \quad k = 1,\ldots,N, \tag{9}$$

with $x[0] = x_0 \in \mathcal{P}^-[0]$ would satisfy $x[k] \in \mathcal{P}^-[k]$, $k = 1,\ldots,N$, whatever are admissible $v[\cdot]$ and $V[\cdot]$. Moreover, introduce a family of such tubes $\mathcal{P}^-[\cdot]$.

Problem 3. Let $u[k] \equiv 0$. Find a polyhedral tube $\mathcal{P}^-[\cdot]$ that satisfies $\mathcal{P}^-[N] = \mathcal{M}$ and $\mathcal{P}^-[k] \subseteq \mathcal{Y}[k]$, $k = 0,\ldots,N-1$, and find a corresponding feedback control strategy $U = U[k,x]$ such that $U[k,x] \in \mathcal{U}[k]$ for $x \in \mathcal{P}^-[k-1]$, $k = 1,\ldots,N$, and each solution $x[\cdot]$ to

$$x[k] = (A[k] + U[k,x[k-1]] + V[k])\,x[k-1] + C[k]v[k], \quad k = 1,\ldots,N, \tag{10}$$

with $x[0] = x_0 \in \mathcal{P}^-[0]$ would satisfy $x[k] \in \mathcal{P}^-[k]$, $k = 1,\ldots,N$, whatever are admissible $v[\cdot]$ and $V[\cdot]$. Moreover, introduce a family of such tubes $\mathcal{P}^-[\cdot]$.

We call the tubes $\mathcal{P}^-[\cdot]$ *polyhedral solvability tubes.*

The paper is organized as follows. In Section 2, we briefly recall the known properties of set operations with parallelotopes and primary polyhedral estimates for the results of such operations, which are used below. Then we recall a scheme, which was proposed earlier in [22]

[3]The terms "internal" and "external" are used here to align with close works (see, for example, [2, 3, 16, 17, 19, 20]). There are also many works, where the other equivalent terms "inner" and "outer" are used (see, for example, [15, 23, 31–33]).

on this base, to solve Problem 2 for the case without the matrix uncertainty. This gives the opportunity to compare this scheme with the new one described in Section 3 for more general case. In Sections 3 and 4, we describe the new unified techniques to solve Problem 2 and Problem 3 respectively. These techniques are extensions of the ones from [24–26] for the more complicated case under state constraints. Both techniques provide control strategies with the attractive property that they can be constructed on the base of the corresponding polyhedral tubes by explicit formulas. In Section 5, illustrative examples are presented.

To distinguish the control strategy described in [22] and in Section 2 from the control strategy described in Section 3 we call the first one as the control of type I, and the second one as the control of type II.

2. Solving Problem 2 for the case without matrix uncertainty (way I)

Here we briefly recall the first method (way I), which was proposed in [22], to solve Problem 2 for the case without the matrix uncertainty (cases (A) and (B,i)).

This method uses properties of set operations with parallelotopes and primary polyhedral estimates for the results of such operations.

For completeness of the exposition, recall some known results about mentioned primary polyhedral estimates. Some of them will be also used in Sections 3 and 4.

Let $\mathcal{P}^k = \mathcal{P}[p^k, \bar{P}^k]$, $k = 1, 2$, $\bar{P}^1 \in \mathbb{R}^{n\times n}$, $\bar{P}^2 \in \mathbb{R}^{n\times m}$. Internal parallelotope-valued estimates for $\mathcal{Q} = \mathcal{P}^1 + \mathcal{P}^2$ can be constructed [20] in the form $\boldsymbol{P}^-_\Gamma(\mathcal{Q}) = \mathcal{P}[p^1{+}p^2,\ \bar{P}^1{+}\bar{P}^2\Gamma]$, where

$$\Gamma \in \mathcal{G}^{m\times n} = \{\Gamma = \{\gamma^\beta_\alpha\} \in \mathbb{R}^{m\times n} \,|\, \|\Gamma\| \le 1\}, \quad \|\Gamma\| = \max_{1\le i\le m} \sum_{j=1}^{n} |\gamma^j_i|.$$

The parameter Γ specify the whole family of estimates.

Let $\det \bar{P}^1 \neq 0$. Then the Minkowski difference $\mathcal{Q} = \mathcal{P}^1 \dot{-} \mathcal{P}^2$ is either a parallelepiped or an empty set [20]: if $\pi^* = \mathrm{e} - \mathrm{Abs}\,((\bar{P}^1)^{-1}\bar{P}^2)\,\mathrm{e} \ge 0$, then $\mathcal{Q} = \mathcal{P}[p^1 - p^2, \bar{P}^1 \operatorname{diag} \pi^*]$; otherwise $\mathcal{Q} = \emptyset$.

Recall the simple way (briefly described in [21]) of constructing for $\mathcal{Q}$ internal estimates $\boldsymbol{P}^-_{v,V}(\mathcal{Q})$ with arbitrary fixed center $v \in \mathcal{Q}$ and orientation matrix $V = \{v^j\} \in \mathbb{R}^{n\times n}$ for the case when the set $\mathcal{Q}$ is a polytope given as the intersection of $\Upsilon \ge n + 1$ strips: $\mathcal{Q} = \bigcap_{j=1}^{\Upsilon} \Sigma^j$, $\Sigma^j = \Sigma(c_j, s^j, \sigma_j) = \{x |\, |(x, s^j) - c_j| \le \sigma_j\}$.

Suppose $v \in \mathcal{Q}$ and $\det V \neq 0$. First we describe how to find $\nu = \nu(v, V) \in \mathbb{R}^n$ that generate the internal estimates $\mathcal{P}^- = \mathcal{P}(v, V, \nu)$ for $\mathcal{Q}$. Let us consider the following system of linear inequalities (where $A = \{a^j_i\} = \{a^j\} \in M^{n\times\Upsilon}$, $b \in \mathbb{R}^\Upsilon$):

$$\begin{gathered} A^\top \nu \le b, \quad \nu \ge 0, \\ a^j_i = |(v^i, s^j)|, \quad i = 1, \dots, n,\ j = 1, \dots, \Upsilon, \\ b_j = \min\{\sigma_j + c_j - (v, s^j), \sigma_j - c_j + (v, s^j)\}, \quad j = 1, \dots, \Upsilon. \end{gathered} \tag{11}$$

Lemma 2.1. *Let $v \in \mathcal{Q}$ and $\det V \neq 0$. Then $a^j \neq 0$, $j = 1, \dots, \Upsilon$, and $A \ge 0$, $b \ge 0$. If the interior $\operatorname{int} \mathcal{Q}$ of $\mathcal{Q}$ is nonempty and $v \in \operatorname{int} \mathcal{Q}$, then $b > 0$.*

Lemma 2.2. *If $v \in \mathcal{Q}$ and $\det V \neq 0$, then $\mathcal{P}(v, V, \nu) \subseteq \mathcal{Q}$ iff ν satisfies (11).*

Lemma 2.3. *Let* $\mathcal{Q} = \bigcap_{j=1}^{\Upsilon} \Sigma^j$ *be the bounded polytope with a nonempty interior;* $v \in \operatorname{int}\mathcal{Q}$*, and* $\det V \neq 0$*. Let vectors* ν^0 *and* ν^* *be determined by the formulas*

$$\nu_i^0 = (1/n)\,\min\{b_j/a_i^j \mid j = 1,\dots,\Upsilon,\ a_i^j \neq 0\}, \quad i = 1,\dots,n, \tag{12}$$

$$\nu^* = \gamma\,\nu^0, \quad \gamma = \min\{b_j/(a^j,\nu^0) \mid j = 1,\dots,\Upsilon,\ (a^j,\nu^0) \neq 0\}. \tag{13}$$

Then $\nu^0 > 0$ *and* $\nu^* > 0$*; both* ν^0 *and* ν^* *satisfy (11), and consequently determine the internal estimates for* $\mathcal{Q}$*, and we have* $\mathcal{P}(v,V,\nu^0) \subseteq \mathcal{P}(v,V,\nu^*) \subseteq \mathcal{Q}$*.*

Note that due to the boundedness of $\mathcal{Q}$, the condition $v \in \operatorname{int}\mathcal{Q}$, and Lemma 2.1, the sets of elements under the signs “min” in (12) and (13) are not empty, and the numerators of all elements are positive. Let us denote $\boldsymbol{P}^-_{v,V}(\mathcal{Q}) = \mathcal{P}(v,V,\nu^*)$.

It was supposed above that the point $v \in \operatorname{int}\mathcal{Q}$ is known. It is not difficult to find such point for some types of sets [21]. Generally, a point $x^* \in \operatorname{int}\mathcal{Q}$ may be found for a fixed V by solving some optimization problem, for example, the following one: find $x^* \in \operatorname{Argmax}\{\operatorname{vol}\boldsymbol{P}^-_{v,V}(\mathcal{Q}) \mid v \in \mathcal{Q}\}$, which may be replaced by minimizing $f(v)$, where $f(v) = -\prod_{i=1}^{n} \nu_i^*$ if $b \geq 0$ and $f(v) = -\sum_{1 \leq j \leq \Upsilon : b_j < 0} b_j$ otherwise (the numerical Nelder-Mead simplex method may be applied).

The following polyhedral analogue of the control synthesis (7), (8) was proposed in [22] on the base of the above primary estimates.

Let us consider the *parametric family* of tubes $\mathcal{P}^-[\cdot]$ which satisfy the following system of recurrent relations:

$$\mathcal{P}^-[N] = \mathcal{M};$$

$$\mathcal{P}^-[k] = \begin{cases} \mathcal{P}^{0-}[k] & \text{if } \mathcal{P}^{0-}[k] \subseteq \mathcal{Y}[k], \\ \boldsymbol{P}^-_{p^-[k],P^-[k]}(\mathcal{P}^{0-}[k] \cap \mathcal{Y}[k]) & \text{otherwise,} \end{cases} \qquad k = N-1,\dots,0. \tag{14}$$

where $\mathcal{P}^{0-}[\cdot] = \mathcal{P}[p^{0-}[\cdot], \bar{P}^{0-}[\cdot]]$ satisfy the relations

$$\mathcal{P}^{0-}[k-1] = A[k]^{-1}\boldsymbol{P}^-_{\Gamma[k]}((\mathcal{P}^-[k] \dot{-} C[k]\mathcal{Q}[k]) - B[k]\mathcal{R}[k]), \quad k = N,\dots,1. \tag{15}$$

Thus, for each time step $k \in \{N-1,\dots,0\}$, first we calculate the parallelotope $\mathcal{P}^{0-}[k]$ using the formulas for the Minkowski difference and for the described primary internal estimate for the Minkowski sum of parallelotopes. Secondly we calculate the internal estimate for the intersection of the constructed parallelotope and a zone. Here matrices $\Gamma[k] \in \mathbb{R}^{n_u \times n}$ and $P^-[k] \in \mathbb{R}^{n \times n}$, and vectors $p^-[k]$ should be such that $\|\Gamma[k]\| \leq 1$, $\det P^-[k] \neq 0$, $p^-[k] \in \operatorname{int}(\mathcal{P}^{0-}[k] \cap \mathcal{Y}[k])$; they serve as admissible parameters of the family of tubes.

If we have solved this system (14), (15) from $k = N$ backwards towards $k = 0$ for fixed admissible parameters $\Gamma[\cdot]$, $P^-[\cdot]$, and $p^-[\cdot]$, then we can construct the following control strategy (the control of type I), which is connected with the obtained polyhedral tube $\mathcal{P}^-[\cdot]$:

$$u[k,x] \in \mathcal{U}^-[k,x] = \mathcal{R}[k] \cap \{u \mid B[k]u \in \mathcal{P}^-[k] \dot{-} C[k]\mathcal{Q}[k] - A[k]x\}, \quad k = 1,\dots,N. \tag{16}$$

Theorem 2.1. *(See [22]). We consider Problem 2 for the system (1), (2), (4), (6) under Assumption 1 for the case without the matrix uncertainty (i. e., either case (A) or* *case (B,i)). In the system (14)–(15), let* $\Gamma[k]$ *be arbitrary matrices that satisfy* $\Gamma[k] \in \mathcal{G}^{n_u \times n}$, $k = N, \ldots, 1$, $P^-[k]$ *be arbitrary nonsingular matrices,* $p^-[k]$ *be arbitrary vectors such that* $p^-[k] \in \operatorname{int}(\mathcal{P}^{0-}[k] \cap \mathcal{Y}[k])$, *and let the sets* $\mathcal{P}^{0-}[k]$ *and* $\mathcal{P}^-[k]$ *turn out to be nondegenerate parallelotopes for all* $k = N-1, \ldots, 0$. *Then the tube* $\mathcal{P}^-[\cdot]$ *and any control strategy* $u[\cdot,\cdot]$ *that satisfies (16) give a particular solution to Problem 2.*

Remark 2.1. To find the value $u[k,x]$ we need to find some point in $\mathcal{U}^-[k,x]$. Since $\mathcal{R}[k]$ are parallelepipeds, it is easy to present this sets $\mathcal{U}^-[k,x]$ in the form of the intersection of several strips: $\bigcap\limits_{i=1}^{\Upsilon} \Sigma^i$, $\Upsilon = n_u + n$. Such point can be found, for example, by a Fejér processes [22].

3. Solving Problem 2 (way II)

Let us consider Problem 2 in general case. Let us introduce the parametric family of tubes $\mathcal{P}^-[\cdot]$ which satisfy the following system of recurrent relations:

$$\mathcal{P}^-[N] = \mathcal{P}[p^-[N], \bar{P}^-[N]] = \mathcal{M}, \quad \text{i.e., } p^-[N] = p_{\mathrm{f}}, \quad \bar{P}^-[N] = \bar{P}_{\mathrm{f}},$$
$$\mathcal{P}^-[k] = \begin{cases} \mathcal{P}^{0-}[k] & \text{if } \mathcal{P}^{0-}[k] \subseteq \mathcal{Y}[k], \\ \boldsymbol{P}^-_{p^-[k],P^-[k]}(\mathcal{P}^{0-}[k] \cap \mathcal{Y}[k]) & \text{otherwise,} \end{cases} \qquad k = N-1, \ldots, 0, \tag{17}$$

where $\mathcal{P}^{0-}[\cdot] = \mathcal{P}[p^{0-}[\cdot], \bar{P}^{0-}[\cdot]]$ satisfy the relations ($k = N, \ldots, 1$)

$$p^{0-}[k-1] = D[k]^{-1}(p^-[k] - B[k]r[k] - C[k]q[k]), \quad D[k] = A[k] + \tilde{V}[k], \tag{18}$$

$$\bar{P}^{0-}[k-1] = D[k]^{-1}(\bar{P}^-[k] \operatorname{diag}(\mathrm{e} - \gamma[k] - \beta[k]) - B[k]\bar{R}[k]\Gamma[k]). \tag{19}$$

$$\gamma[k] = (\operatorname{Abs}(\bar{P}^-[k]^{-1} C[k] \bar{Q}[k]))\,\mathrm{e}, \tag{20}$$

$$\beta[k] = \max_{z \in \mathbb{E}(\mathcal{P}^{0-}[k-1])} (\operatorname{Abs}(\bar{P}^-[k]^{-1}))\hat{V}[k] \operatorname{Abs} z. \tag{21}$$

Here the symbol $\mathbb{E}(\mathcal{P})$ denotes the set of all vertices of $\mathcal{P} = \mathcal{P}[p, \bar{P}]$ (i.e., the set of points $p + \sum\limits_{i=1}^{m} \bar{p}^i \xi_i$ with $\xi_i \in \{-1, 1\}$); the operation of maximum is understood component-wise. Thus, for each time step $k \in \{N-1, \ldots, 0\}$, first we calculate the parallelotope $\mathcal{P}^{0-}[k]$ using the relations for centers $p^{0-}[k]$ and matrices $\bar{P}^{0-}[k]$ of parallelotopes, where the right-hand sides in (18), (19) are determined by the explicit formulas except vectors $\beta[k]$. These vectors $\beta[k]$ satisfy the systems of nonlinear equations (21) which can be written in the following form:

$$\beta[k] = H[k, \beta[k]],$$
$$H[k,\beta] = \max_{\xi \in \mathbb{E}(\mathcal{C})} (\operatorname{Abs}(\bar{P}^-[k]^{-1}))\hat{V}[k] \operatorname{Abs}(p^{0-}[k-1] +$$
$$+ D[k]^{-1}((\bar{P}^-[k] \operatorname{diag}(\mathrm{e} - \gamma[k]) - B[k]\bar{R}[k]\Gamma[k])\xi - \bar{P}^-[k] \operatorname{diag} \xi \cdot \beta), \tag{22}$$

where $\mathcal{C} = \mathcal{P}(0, I, \mathrm{e})$. For cases (A) and (B,i), the situation is simplified and we have $\beta[k] = 0$. Secondly we calculate the internal estimate for $\mathcal{P}^{0-}[k] \cap \mathcal{Y}[k]$ similarly to (14).

Here matrices $\Gamma[k] \in \mathbb{R}^{n_u \times n}$ and $P^-[k] \in \mathbb{R}^{n \times n}$, and vectors $p^-[k]$ play the same role as in Section 2 and serve as admissible parameters of the family of tubes.

Let us consider the following control strategy (the control of type II):

$$u[k,x] = r[k] + \lambda[k,x]\, \bar{R}[k]\Gamma[k]\bar{P}^{0-}[k-1]^{-1}(x - p^{0-}[k-1]),$$
$$\lambda[k,x] = \min\{1, \|\Gamma[k]\bar{P}^{0-}[k-1]^{-1}(x - p^{0-}[k-1])\|_\infty^{-1}\}, \quad k = 1, \ldots, N. \tag{23}$$

Theorem 3.1. *We consider Problem 2 for the system (1), (2), (4)–(6) under Assumption 1. In the system (17)–(21), let* $\Gamma[k]$ *be arbitrary matrices that satisfy* $\Gamma[k] \in \mathcal{G}^{n_u \times n}$, $k = N, \ldots, 1$, $P^-[k]$ *be arbitrary nonsingular matrices,* $p^-[k]$ *be arbitrary vectors such that* $p^-[k] \in \mathrm{int}\,(\mathcal{P}^{0-}[k] \cap \mathcal{Y}[k])$*; let the following inequalities* $\mathrm{e} - \gamma[k] - \beta[k] \geq 0$ *be satisfied, and the sets* $\mathcal{P}^{0-}[k]$ *and* $\mathcal{P}^-[k]$ *turn out to be nondegenerate parallelotopes for all* $k = N-1, \ldots, 0$. *Then the tube* $\mathcal{P}^-[\cdot] = \mathcal{P}[p^-[\cdot], \bar{P}^-[\cdot]]$ *and the control strategy (23) give a particular solution to Problem 2.*

The scheme of the proof is the development of the one from [26, Theorem 3] and is similar to the proof of Theorem 4.1 (see below). Note that $u[k,x] \in \mathcal{R}[k]$ for any $x \in \mathbb{R}^n$, and it follows from the proof that if we start from $x_0 \in \mathcal{P}^-[0]$ and apply (23), then we obtain $u[k, x[k-1]]$ with $\lambda[k, x[k-1]] = 1$ for all $k = 1, \ldots, N$.

Remark 3.1. It is not difficult to see that for cases (A) and (B,i) (without matrix uncertainty) we have $\beta[k] = 0$, $k = 1, \ldots, N$, and formulas (18)–(21) for calculating the parallelotopes $\mathcal{P}^{0-}[k]$ coincide with the ones from (15). Thus in these cases the families of tubes $\mathcal{P}^-[\cdot]$ from Theorem 2.1 and Theorem 3.1 are the same, but control strategies (16) and (23) are different. The attractive property of the controls of type II is their explicit form.

Remark 3.2. It follows from the contraction operator principle [35, p.319] that if the operator $H[k,\beta]$ in (22) is contractive, i. e., $\|H[k,\beta^1] - H[k,\beta^2]\|_\infty \leq L\|\beta^1 - \beta^2\|_\infty$ for any $\beta^1, \beta^2 \in \mathbb{R}^n$, where $L = L[k] \in (0,1)$, then the equation $\beta = H[k,\beta]$ has the unique solution $\beta = \beta[k]$ (it is nonnegative for our operator H), which can be found by the simple iteration $\beta^{l+1} = H[k,\beta^l]$, $l = 0, 1, \ldots$, starting from arbitrary β^0; if $\beta^0 = 0$, then we have $\|\beta^l - \beta\|_\infty \leq L^l(1-L)^{-1}\|(\mathrm{Abs}\,(\bar{P}^-[k]^{-1}))\hat{V}[k]\|\,(\|p^{0-}[k-1]\|_\infty + \|P^1[k]\|)$, where $P^1[k] = D[k]^{-1}(\bar{P}^-[k]\mathrm{diag}\,(\mathrm{e} - \gamma[k]) - B[k]\bar{R}[k]\Gamma[k])$.

Remark 3.3. Let the system (1)–(5) be obtained by the Euler approximations of a similar differential system so that $A[k] = I + h_N A(t_{k-1})$, $\tilde{V}[k] = h_N \tilde{V}(t_{k-1})$, $\hat{V}[k] = h_N \hat{V}(t_{k-1})$, $B[k] = h_N B(t_{k-1})$, $\mathcal{R}[k] = \mathcal{R}(t_{k-1})$, $C[k] = h_N C(t_{k-1})$, $\mathcal{Q}[k] = \mathcal{Q}(t_{k-1})$, $t_k = kh_N \in [0,\theta]$, $h_N = \theta N^{-1}$. Consider a fixed k. If $\det \bar{P}^-[k] \neq 0$ and the discretization step h_N is sufficient small, then the operator $H[k,\beta]$ turns out to be contractive and we have the desired inequality $\mathrm{e} - \gamma[k] - \beta[k] > 0$ for this k. However, this does not generally imply the existence of nonempty parallelotopes $\mathcal{P}^-[k]$ with nonsingular matrices $\bar{P}^-[k]$ for all $k = N, \ldots, 1$, since the value of such “small” h_N depends on k.

Remark 3.4. Theorem 3.1 describes the whole parametric family of tubes $\mathcal{P}^-[\cdot]$ for a fixed $N > 0$. The set $\mathcal{W}^-_{0,N} = \bigcup \mathcal{P}^-[0]$, where the union is taken over all tubes $\mathcal{P}^-[\cdot]$ satisfying Theorem 3.1, is a subset of the set of all initial positions which can be steered to the terminal set $\mathcal{M}$ during the time N by solving Problem 2. Also we have $\mathcal{P}^-[0] \subseteq \mathcal{W}[0]$ for any mentioned tube $\mathcal{P}^-[\cdot]$ and $\mathcal{W}^-_{0,N} \subseteq \mathcal{W}[0]$, where $\mathcal{W}[0]$ is the solvability set for Problem 1.

Remark 3.5. Let us also indicate some connections between the solutions to Problem 1 and Problem 2 and results from viability theory [6]. Let system (1), (2), (4)–(6) without uncertainty (case (A)) be time-invariant (i. e., the coefficients and the sets $\mathcal{R}$ and $\mathcal{Y}$ do not depend on k), $\mathcal{M} \subseteq \mathcal{Y}$, and $k = 1, 2, \ldots$. According to [6, p.71], the subset $\mathrm{Capt}\,(\mathcal{Y}, \mathcal{M})$ of initial states $x_0 \in \mathcal{Y}$ such that at least one solution to this system starting at x_0 is viable in $\mathcal{Y}$ (i. e., (6) is satisfied for some N) until it reaches $\mathcal{M}$ in finite time is called the *capture basin of* $\mathcal{M}$ *viable in* $\mathcal{Y}$. If we denote by $\mathcal{W}_N[0]$ the solvability set $\mathcal{W}[0]$ for Problem 1 formulated for a fixed final time N, then we obviously obtain $\bigcup_{N \geq 0} \mathcal{W}_N[0] = \mathrm{Capt}\,(\mathcal{Y}, \mathcal{M})$. If we fix some $N > 0$ and solve the corresponding Problem 2, then $\mathcal{P}^-[0] \subseteq \mathrm{Capt}\,(\mathcal{Y}, \mathcal{M})$. Therefore the set $\mathcal{W}^-_{0,N}$ from Remark 3.4 is also some internal estimate for $\mathrm{Capt}\,(\mathcal{Y}, \mathcal{M})$: $\mathcal{W}^-_{0,N} \subseteq \mathrm{Capt}\,(\mathcal{Y}, \mathcal{M})$, and also $\bigcup_{N \geq 0} \mathcal{W}^-_{0,N} \subseteq \mathrm{Capt}\,(\mathcal{Y}, \mathcal{M})$.

4. Solving Problem 3

Now let us consider Problem 3 with controls in the matrices.

Let us introduce the parametric family of tubes $\mathcal{P}^-[\cdot]$ which satisfy the following system of recurrent relations:

$$\mathcal{P}^-[N] = \mathcal{P}[p^-[N], \bar{P}^-[N]] = \mathcal{M}, \quad \text{i. e., } p^-[N] = p_{\mathrm{f}}, \quad \bar{P}^-[N] = \bar{P}_{\mathrm{f}},$$
$$\mathcal{P}^-[k] = \begin{cases} \mathcal{P}^{0-}[k] & \text{if } \mathcal{P}^{0-}[k] \subseteq \mathcal{Y}[k], \\ \boldsymbol{P}^-_{p^-[k],P^-[k]}(\mathcal{P}^{0-}[k] \cap \mathcal{Y}[k]) & \text{otherwise,} \end{cases} \qquad k = N{-}1, \ldots, 0, \tag{24}$$

where $\mathcal{P}^{0-}[\cdot] = \mathcal{P}[p^{0-}[\cdot], \bar{P}^{0-}[\cdot]]$ satisfy the relations ($k = N, \ldots, 1$)

$$p^{0-}[k{-}1] = D[k]^{-1}(p^-[k] - C[k]q[k]), \quad D[k] = A[k] + \tilde{U}[k] + \tilde{V}[k], \tag{25}$$

$$\bar{P}^{0-}[k{-}1] = H[k, \bar{P}^{0-}[k{-}1]], \quad k = N, \ldots, 1, \tag{26}$$

$$H[k, P] = (D[k] - \operatorname{diag}\alpha[k, P])^{-1}\bar{P}^-[k] \operatorname{diag}(\mathrm{e} - \beta[k, P] - \gamma[k]), \tag{27}$$

$$\alpha_i[k, P] = \alpha_i[k, P; J[k]] = \hat{u}_i^{j_i}[k]\, \eta_{j_i}[k, P]\, (\mathrm{e}^{i\top}(\operatorname{Abs} P)\, \mathrm{e})^{-1}, \quad i = 1, \ldots, n, \tag{28}$$

$$\eta[k, P] = \max\{0, \operatorname{Abs} p^{0-}[k{-}1] - (\operatorname{Abs} P)\mathrm{e}\}, \tag{29}$$

$$\beta[k, P] = \max\{\operatorname{Abs}(\bar{P}^-[k]^{-1})\, \hat{V}[k] \operatorname{Abs}(p^{0-}[k{-}1] + P\xi) \mid \xi \in \mathbb{E}(\mathcal{C})\}, \tag{30}$$

$$\gamma[k] = (\operatorname{Abs}(\bar{P}^-[k]^{-1}C[k]\bar{Q}[k]))\mathrm{e}, \quad k = N, \ldots, 1. \tag{31}$$

Here the centers of parallelotopes $\mathcal{P}^{0-}[k]$ are determined by system of explicit recurrent relations (25) while the matrices $\bar{P}^{0-}[k]$ are determined by the relations (26)–(31), where, for any step k, we need to solve a system of nonlinear equations $P = H[k, P]$ with respect to the unknown matrix $P = \bar{P}^{0-}[k{-}1]$. Formulas for $\alpha_i[k, P]$ contain $J[k]$, where $J[k] = \{j_1[k], \ldots, j_n[k]\}$ are arbitrary permutations of natural numbers $\{1, \ldots, n\}$. Thus, we have again the parametric family of polyhedral tubes, where the parameters are this vector function $J[\cdot]$ and also the matrix function $P^-[\cdot]$ and the vector function $p^-[\cdot]$. The last two parameters, which appear in formulas (24) for internal estimates for intersections of parallelotopes with zones, are the same as in Sections 2 and 3 for the solutions to Problem 2.

Again, if we have solved the above system from $k = N$ backwards towards $k = 0$ for fixed admissible parameters, then we can construct by explicit formulas the following control strategy, which is connected with the obtained pair of polyhedral tubes $\mathcal{P}^-[\cdot]$ and $\mathcal{P}^{0-}[\cdot]$:

$$\mathrm{e}^{i\top}U[k, x] = \begin{cases} \mathrm{e}^{i\top}\tilde{U}[k] - \alpha_i[k, \bar{P}^{0-}[k{-}1]](x_i - p_i^{0-}[k{-}1])(x_{j_i})^{-1}\mathrm{e}^{j_i\top} & \text{if } x_{j_i} \neq 0, \\ \mathrm{e}^{i\top}\tilde{U}[k] \quad \text{if } x_{j_i} = 0, \quad i = 1, \ldots, n. \end{cases} \tag{32}$$

First of all note that the control strategy (32) has the following property.

Lemma 4.1. *If* $\det \bar{P}^{0-}[k{-}1] \neq 0$ *and* $\mathcal{P}^-[k{-}1] \subseteq \mathcal{P}^{0-}[k{-}1]$*, then the control strategy* $U[k, x]$ *from (32) acts on* $x \in \mathcal{P}^-[k{-}1]$ *according to the following rule:*

$$U[k, x]\, x = \tilde{U}[k]\, x - \operatorname{diag}\alpha[k, \bar{P}^{0-}[k{-}1]]\, (x - p^{0-}[k{-}1]) \quad (\textit{for } x \in \mathcal{P}^-[k{-}1]). \tag{33}$$

Proof. If the matrix $\bar{P}^{0-}[k-1]$ is nonsingular, then it can not have zero rows. Therefore the denominators of all the components of $\alpha[k, \bar{P}^{0-}[k-1]; J[k]]$ are different from zero: ${\mathrm{e}^i}^\top(\mathrm{Abs}\,\bar{P}^{0-}[k-1])\,\mathrm{e} \neq 0$. Obviously, if $x_{j_i} \neq 0$, then both formulas (32) and (33) give the same result for ${\mathrm{e}^i}^\top U[k,x]\,x$. Let $x_{j_i} = 0$. Let us consider two possible cases for values of $\eta_{j_i}[k, \bar{P}^{0-}[k-1]]$. If $\eta_{j_i}[k, \bar{P}^{0-}[k-1]] = 0$, then $\alpha_i[k, \bar{P}^{0-}[k-1]] = 0$ due to (28), and both formulas (32) and (33) give the same result for ${\mathrm{e}^i}^\top U[k,x]\,x$ again. Now suppose that $\eta_{j_i}[k, \bar{P}^{0-}[k-1]] > 0$, i. e., $|p^{0-}_{j_i}[k-1]| - (\mathrm{e}^{j_i})^\top(\mathrm{Abs}\,\bar{P}^{0-}[k-1])\mathrm{e} > 0$. It follows from the inclusions $x \in \mathcal{P}^-[k-1] \subseteq \mathcal{P}^{0-}[k-1]$ that $x = p^{0-}[k-1] + \bar{P}^{0-}[k-1]\zeta^0$, where $\|\zeta^0\|_\infty \leq 1$. Thus $|x_{j_i}| \geq |p^{0-}_{j_i}[k-1]| - |(\mathrm{e}^{j_i})^\top \bar{P}^{0-}[k-1]\zeta^0| \geq |p^{0-}_{j_i}[k-1]| - (\mathrm{e}^{j_i})^\top(\mathrm{Abs}\,\bar{P}^{0-}[k-1])\mathrm{e} > 0$, i. e., in the considered case the equality $x_{j_i} = 0$ is impossible. ■

Theorem 4.1. *We consider Problem 3 for the system (1), (3), (4)–(6) under Assumption 1. Let* $J[k] = \{j_1[k], \ldots, j_n[k]\}$, $k = N, \ldots, 1$, *be arbitrary permutations of natural numbers* $\{1, \ldots, n\}$ *and the system (24)–(31) has a solution* $(p^{0-}[\cdot], \bar{P}^{0-}[\cdot], p^-[\cdot], \bar{P}^-[\cdot])$ *that satisfy the following relations:*

$$\mathrm{e} - \beta[k, \bar{P}^{0-}[k-1]] - \gamma[k] > 0, \quad k = N, \ldots, 1, \tag{34}$$

$$\det \bar{P}^{0-}[k] \neq 0,\ \det \bar{P}^-[k] \neq 0,\ p^-[k] \in \mathrm{int}\,(\mathcal{P}^{0-}[k] \cap \mathcal{Y}[k]), \quad k = N-1, \ldots, 0. \tag{35}$$

Then the tube $\mathcal{P}^-[\cdot] = \mathcal{P}[p^-[\cdot], \bar{P}^-[\cdot]]$ *and the control strategy (32) give a particular solution to Problem 3.*

Proof. Let $\mathcal{P}^{0-}[k]$ and $\mathcal{P}^-[k]$, $k = N, \ldots, 0$, that satisfy (34), (35) be found from (24)–(31). Inclusions $\mathcal{P}^-[k] \subseteq \mathcal{Y}[k]$ follow from (24) and Section 2.

Let $x[\cdot]$ be the solution of (10) that corresponds to $x[0] = x_0 \in \mathcal{P}^-[0]$ (where $x_0 = p^-[0] + \bar{P}^-[0]\zeta_0$, $\|\zeta_0\|_\infty \leq 1$), to the control strategy $U = U[k,x]$ from (32), and to arbitrary admissible $v[\cdot]$ and $V[\cdot] = \tilde{V}[\cdot] + \Delta V[\cdot]$ (i. e., $v[k] = q[k] + \bar{Q}[k]\chi[k]$, $\|\chi[k]\|_\infty \leq 1$; $\mathrm{Abs}\,(\Delta V[k]) \leq \hat{V}[k]$ for all k). Let us represent vectors $x[k]$ in the form

$$x[k] = p^-[k] + \bar{P}^-[k]\zeta[k] = p^{0-}[k] + \bar{P}^{0-}[k]\zeta^0[k], \quad k = 0, \ldots, N. \tag{36}$$

First we prove by induction that if $x = x[k-1] \in \mathcal{P}^-[k-1]$ (therefore $\|\zeta^0[k-1]\|_\infty \leq 1$ due to $x[k-1] \in \mathcal{P}^-[k-1] \subseteq \mathcal{P}^{0-}[k-1]$), then $\|\zeta[k]\|_\infty \leq 1$, i. e., $x[k] \in \mathcal{P}^-[k]$.

Let we already have $x[k-1] \in \mathcal{P}^-[k-1]$. Using (24), we have $x[k-1] \in \mathcal{P}^-[k-1] \subseteq \mathcal{P}^{0-}[k-1]$. Then there exist $\zeta[k-1]$ and $\zeta^0[k-1]$ such that

$$\begin{gathered} x[k-1] = p^-[k-1] + \bar{P}^-[k-1]\zeta[k-1] = p^{0-}[k-1] + \bar{P}^{0-}[k-1]\zeta^0[k-1], \\ \mathrm{Abs}\,\zeta[k-1] \leq \mathrm{e}, \quad \mathrm{Abs}\,\zeta^0[k-1] \leq \mathrm{e}. \end{gathered} \tag{37}$$

It follows from (36), (10), (37), and (4) that

$$\zeta[k] = \bar{P}^-[k]^{-1}(x[k] - p^-[k]) =$$

$$= \bar{P}^-[k]^{-1}((D[k] + U[k, x[k-1]] - \tilde{U}[k] + \Delta V[k])\,x[k-1] + C[k]v[k] - p^-[k]) =$$

$$= \bar{P}^-[k]^{-1}(D[k](p^{0-}[k-1] + \bar{P}^{0-}[k-1]\zeta^0[k-1]) + (U[k, x[k-1]] - \tilde{U}[k])\,x[k-1] +$$

$$+ \Delta V[k]x[k-1] + C[k](q[k] + \bar{Q}[k]\chi[k]) - p^-[k]).$$

Using the equality
$$D[k]p^{0-}[k-1] + C[k]q[k] - p^-[k] = 0,$$
which follows from (25), we obtain
$$\zeta[k] = \bar{P}^-[k]^{-1}(D[k]\bar{P}^{0-}[k-1]\zeta^0[k-1] + (U[k,x[k-1]] - \tilde{U}[k])\,x[k-1]) + c[k,x[k-1]], \quad (38)$$
where
$$c[k,x] = \bar{P}^-[k]^{-1}\Delta V[k]\,x + \bar{P}^-[k]^{-1}C[k]\bar{Q}[k]\chi[k].$$
According to (26), (27), we have
$$(D[k] - \mathrm{diag}\,\alpha[k,\bar{P}^{0-}[k-1]])\bar{P}^{0-}[k-1] = \bar{P}^-[k]\,\mathrm{diag}\,(\mathrm{e} - \beta[k,\bar{P}^{0-}[k-1]] - \gamma[k]).$$
Taking into account the last equality and formula (33) from Lemma 4.1, and also (37), we can conclude from (38) that
$$\zeta[k] = \bar{P}^-[k]^{-1}(D[k]\bar{P}^{0-}[k-1]\zeta^0[k-1] - \mathrm{diag}\,\alpha[k,\bar{P}^{0-}[k-1]](x[k-1]-p^{0-}[k-1]))+$$
$$+c[k,x[k-1]] = \bar{P}^-[k]^{-1}(D[k] - \mathrm{diag}\,\alpha[k,\bar{P}^{0-}[k-1]])\,\bar{P}^{0-}[k-1]\zeta^0[k-1] + c[k,x[k-1]] =$$
$$= \mathrm{diag}\,(\mathrm{e} - \beta[k,\bar{P}^{0-}[k-1]] - \gamma[k])\,\zeta^0[k-1] + c[k,x[k-1]]. \quad (39)$$
Using the relations $x[k-1] \in \mathcal{P}^-[k-1] \subseteq \mathcal{P}^{0-}[k-1]$, we obtain the following estimate for $c[k,x[k-1]]$:
$$\mathrm{Abs}\,c[k,x[k-1]] \le \mathrm{Abs}\,(\bar{P}^-[k]^{-1})\,\hat{V}[k] \max_{z\in\mathcal{P}^-[k-1]} \mathrm{Abs}\ z + (\mathrm{Abs}\,(\bar{P}^-[k]^{-1}C[k]\bar{Q}[k]))\,\mathrm{e} \le$$
$$\le \mathrm{Abs}\,(\bar{P}^-[k]^{-1})\,\hat{V}[k] \max_{z\in\mathcal{P}^{0-}[k-1]} \mathrm{Abs}\ z + \gamma[k] =$$
$$= \beta[k,\bar{P}^{0-}[k-1]] + \gamma[k] \quad \text{for} \quad x[k-1] \in \mathcal{P}^-[k-1]. \quad (40)$$
To obtain the last equality we have also used the following fact. If $a \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{n\times n}$, $B \ge 0$, $\phi(\xi) = B\,\mathrm{Abs}\,(a + A\xi)$, where $\xi \in \mathbb{R}^n$, then $\max\limits_{\xi\in\mathcal{C}} \phi(\xi) = \max\limits_{\xi\in\mathbb{E}(\mathcal{C})} \phi(\xi)$, where the maximum is component-wise. This follows from the convexity of any component of the function ϕ and the fact that $\mathcal{C}$ coincides with the convex hull of its extreme points $\xi^j \in \mathbb{E}(\mathcal{C})$.

From (39), (40), (34), and (37) we conclude that
$$|\zeta_i[k]| \le |1 - \beta_i[k,\bar{P}^{0-}[k-1]] - \gamma_i[k])| \cdot |\zeta_i^0[k-1]| + \beta_i[k,\bar{P}^{0-}[k-1]] + \gamma_i[k] \le$$
$$\le (1 - \beta_i[k,\bar{P}^{0-}[k-1]] - \gamma_i[k])) \cdot 1 + \beta_i[k,\bar{P}^{0-}[k-1]] + \gamma_i[k] = 1, \quad i = 1,\dots,n,$$
and consequently $\mathrm{Abs}\,\zeta[k] \le \mathrm{e}$, i. e., $x[k] \in \mathcal{P}^-[k]$ indeed.

It remains to prove that if $x \in \mathcal{P}^-[k-1]$, then $\mathrm{Abs}\,(U[k,x] - \tilde{U}[k]) \le \hat{U}[k]$. Let us show that $\mathrm{e}^{i\top}\mathrm{Abs}\,(U[k,x]-\tilde{U}[k]) \le \mathrm{e}^{i\top}\hat{U}[k]$ for every fixed $i \in \{1,\dots,n\}$. If x is such that $x_{j_i} = 0$, then the above inequality is satisfied because (32) yields $\mathrm{e}^{i\top}\mathrm{Abs}\,(U[k,x]-\tilde{U}[k]) = 0^\top$ (where $0 \in \mathbb{R}^n$). If $x \in \mathcal{P}^-[k-1]$ and $x_{j_i} \ne 0$, then, according to (32), it is sufficient to show that we have $|\delta_i[k,x]| \le \hat{u}_i^{j_i}[k]$ for such x, where $\delta_i[k,x] = \alpha_i[k,\bar{P}^{0-}[k-1]]\,(x_i - p_i^{0-}[k-1])(x_{j_i})^{-1}$. The inclusions $x \in \mathcal{P}^-[k-1] \subseteq \mathcal{P}^{0-}[k-1]$ imply that we have $x - p^{0-}[k-1] = \bar{P}^{0-}[k-1]\zeta^0$, where $\mathrm{Abs}\,\zeta^0 \le \mathrm{e}$. Therefore $x_{j_i} = p_{j_i}^{0-}[k-1] + (\mathrm{e}^{j_i})^\top(x - p^{0-}[k-1]) = p_{j_i}^{0-}[k-1] + (\mathrm{e}^{j_i})^\top\bar{P}^{0-}[k-1]\zeta^0$ and
$$|\delta_i[k,x]| = \frac{\hat{u}_i^{j_i}[k]\ \max\{0, |p_{j_i}^{0-}[k-1]| - (\mathrm{e}^{j_i})^\top(\mathrm{Abs}\,\bar{P}^{0-}[k-1])\mathrm{e}\} \cdot |(\mathrm{e}^i)^\top\bar{P}^{0-}[k-1]\zeta^0|}{(\mathrm{e}^i)^\top(\mathrm{Abs}\,\bar{P}^{0-}[k-1])\mathrm{e} \cdot |p_{j_i}^{0-}[k-1] + (\mathrm{e}^{j_i})^\top\bar{P}^{0-}[k-1]\zeta^0|}.$$

If $|p^{0-}_{j_i}[k-1]| < (\mathrm{e}^{j_i})^\top(\mathrm{Abs}\,\bar{P}^{0-}[k-1])\mathrm{e}$, then $|\delta_i[k,x]| = 0 \le \hat{u}^{j_i}_i[k]$. Otherwise we have the desired inequality again because

$$|\delta_i[k,x]| \le \frac{\hat{u}^{j_i}_i[k]\,(|p^{0-}_{j_i}[k-1]| - (\mathrm{e}^{j_i})^\top(\mathrm{Abs}\,\bar{P}^{0-}[k-1])\mathrm{e})\cdot(\mathrm{e}^{i})^\top(\mathrm{Abs}\,\bar{P}^{0-}[k-1])\mathrm{e}}{(\mathrm{e}^{i})^\top(\mathrm{Abs}\,\bar{P}^{0-}[k-1])\mathrm{e}\cdot(|p^{0-}_{j_i}[k-1]| - (\mathrm{e}^{j_i})^\top(\mathrm{Abs}\,\bar{P}^{0-}[k-1])\mathrm{e})} = \hat{u}^{j_i}_i[k].$$

Here we used the elementary inequations of the following types: $\max\{0,a\} \le a$ for $a \ge 0$, $|a+b| \ge |a| - |b|$, and $|(\mathrm{e}^i)^\top P\zeta| \le (\mathrm{e}^i)^\top(\mathrm{Abs}\,P)\mathrm{e}$ for ζ such that $\mathrm{Abs}\,\zeta \le \mathrm{e}$, and then we reduced the obtained fraction. ■

Remark 4.1. The remarks similar to Remarks 3.2, 3.3, and [26, Remark 8], concerning the situation with contractive operators H, may be formulated.

Remark 4.2. A remark similar to Remark 3.5 concerning connections between solutions to Problem 3 for time-invariant systems without uncertainty and the related Capt $(\mathcal{Y},\mathcal{M})$ may be formulated.

Remark 4.3. With an unsuccessful choice of admissible parameters in the formulas for the polyhedral tubes from Theorems 2.1, 3.1, and 4.1, a case when at some time step k we can obtain the empty set $\mathcal{P}^-[k]$ is not excluded and therefore we can not construct the solution of the problem under consideration using such parameters.

Remark 4.4. The described tubes $\mathcal{P}^-[\cdot]$ that correspond to different values of admissible parameters can be calculated independently of each other. This allows a natural parallelization of computations. Numerical simulation results testify for efficiency of parallelization for the case without uncertainty and state constraints. Otherwise, the situation is worse (see Remark 4.3). Issues of parallel computations for constructing solvability tubes using ellipsoidal techniques are discussed in [3, 18].

5. Examples

Let us consider two illustrating examples of polyhedral control synthesis for systems obtained by discretization of some differential ones determined on time interval $[0,\theta]$.

Example 5.1. Let Problem 2 be considered and $A \equiv I + \tau\begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}$; $\tilde{V} \equiv 0$; $\hat{V} \equiv 0$ or $\hat{V} \equiv \tau\begin{bmatrix} 0 & 0 \\ 0.1 & 0 \end{bmatrix}$; $B \equiv C \equiv \tau I$; $\mathcal{R} \equiv \mathcal{P}(0, I, (0,1)^\top)$; $\mathcal{Q} \equiv \mathcal{P}(0, I, 0)$ or $\mathcal{Q} \equiv \mathcal{P}(0, I, (0.2, 0)^\top)$; $\mathcal{M} = \mathcal{P}((-0.5, 0)^\top, I, (0.5, 0.5)^\top)$; $\tau = \theta/N$; $\theta = 2$; $N = 200$.

We use the following values of parameters of the tubes: $\Gamma[k]$ are constructed by special formulas similar to [20]; $P^-[k] = P^{0-}[k]$, $k = N-1, \ldots, 0$; $p^-[k]$ are constructed by the Nelder-Mead method to maximize $\mathrm{vol}\,\boldsymbol{P}^-_{p^-[k],P^-[k]}(\mathcal{P}^{0-}[k] \cap \mathcal{Y}[k])$ under above $P^-[k]$.

In Figures 1 and 2, we present results for two cases, namely, case (A) without uncertainty and state constraints and case (B,ii;SC) under both additive and matrix uncertainties and under state constraints. We consider the state constraints of the form $|x_1 + 0.2| \le 0.8$, $|x_2| \le 2.1$, and for simulations we put $v[\cdot]$ similar to [20] and $V[\cdot] \equiv \tilde{V}[\cdot] + \hat{V}[\cdot]$. Target sets $\mathcal{M}$ here and below are shown by dashed lines, state constraints are shown by dash-dot lines.

In the left part of Figure 1, which corresponds to case (A), we present cross-sections $\mathcal{P}^-[0]$ for several tubes $\mathcal{P}^-[\cdot]$ from Theorem 3.1 (Theorem 2.1) and trajectories, which correspond to two types of controls (dash lines for controls of type I and solid lines for controls of type II) for two initial points $x_0 = (-0.3, 2)^\top$ and $x_0 = (-0.45,\ 1.33)^\top$. In the right part of Figure 1, we show some tube $\mathcal{P}^-[\cdot]$ and the corresponding controlled trajectory.

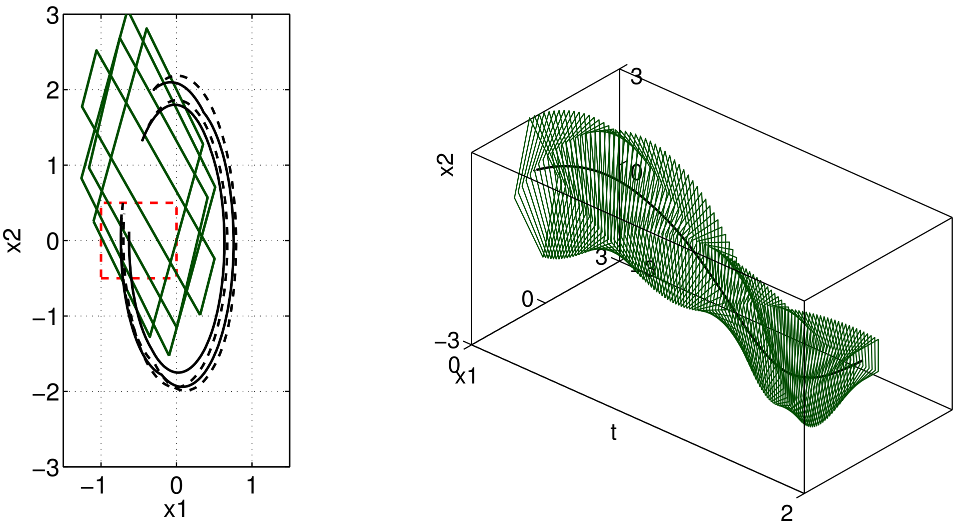

Fig. 1. Example 5.1, case (A): cross-sections $\mathcal{P}^-[0]$ for several tubes $\mathcal{P}^-[\cdot]$ and controlled trajectories of two types for two initial points (left); some tube $\mathcal{P}^-[\cdot]$ and the corresponding controlled trajectory (right)

Figure 2, which corresponds to case (B,ii;SC), is similar to the previous one, but here, in the right part, we show also several cross-sections $\mathcal{P}^-[k]$ of the tube $\mathcal{P}^-[\cdot]$ in the phase plane.

We see that all the presented trajectories reach the target set. Note that if $x_0 \notin \mathcal{P}^-[0]$, then there is no guarantee that the trajectory can be steered, by the corresponding control, into $\mathcal{M}$ satisfying state constraints under any disturbances. For the case (B,ii;SC) with $x_0 = (-0.3, 2)^\top \notin \mathcal{P}^-[0]$, we obtained that that the target set is reached, but the state constraints are slightly violated.

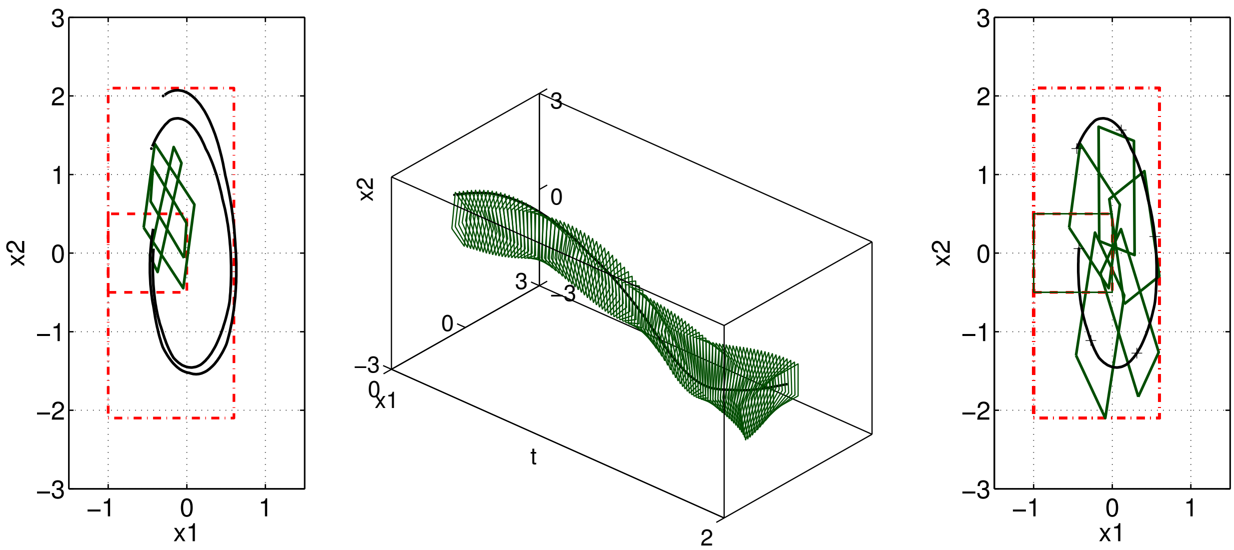

Fig. 2. Example 5.1, case (B,ii;SC): cross-sections $\mathcal{P}^-[0]$ for several tubes $\mathcal{P}^-[\cdot]$ from Theorem 3.1 and controlled trajectories for two initial points (left); some tube $\mathcal{P}^-[\cdot]$ (middle); several cross-sections $\mathcal{P}^-[k]$ for this $\mathcal{P}^-[\cdot]$ and the corresponding controlled trajectory (right)

Example 5.2. Let Problem 3 be considered and $A \equiv I + \tau \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}$; $\tilde{U} \equiv \tau \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$; $\hat{U} \equiv \tau \begin{bmatrix} 0 & 1.5 \\ 0 & 0 \end{bmatrix}$; $\tilde{V} \equiv \tau \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$; $\hat{V} \equiv 0$ or $\hat{V} \equiv \tau \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \end{bmatrix}$; $\mathcal{Q} \equiv \mathcal{P}(0, I, 0)$ or $\mathcal{Q} \equiv \mathcal{P}(0, I, (0.05, 0.05)^\top)$; $C \equiv \tau \cdot I$; $\mathcal{M} = \mathcal{P}((1,1)^\top, I, (0.1, 0.1)^\top)$; $\tau = \theta/N$; $\theta = 0.25$; $N = 200$. State constraints, if any, are determined by the strip $|x_1 - 0.75| \leq 0.35$.

We use Theorem 4.1 with the following values of parameters of the tubes: $J[k]$ are either constant or are constructed using arguments of a "local" volume optimization; $P^-[k]$ and $p^-[k]$ are constructed similarly to Example 5.1.

In Figure 3, we can compare the results of modeling for the following three cases: the case without uncertainties and state constraints ((A), left pictures), and two cases under state constraints, namely, without uncertainties ((A;SC), middle pictures) and with uncertainties in both matrix and additive terms ((B,ii;SC), right pictures). The polyhedral solvability sets for the cases with state constraints and uncertainties turn out to be smaller than for the first one. For each case, we calculated three tubes $\mathcal{P}^-[\cdot]$ that correspond to three parameters $J[\cdot]$, where $J[k] \equiv \{1, 2\}$, $J[k] \equiv \{2, 1\}$, and $J[k]$ are constructed using arguments of a "local" volume optimization; the second and the third tubes visually coincide; the first tube for case (B,ii;SC) could not be calculated up to $k = 0$. At the top of Figure 3, we show the set $\mathcal{M}$ (dashed lines), state constraints (dash-dot lines), parallelotopes $\mathcal{P}^-[0]$ obtained, and controlled trajectories for the initial point $x_0 = (0.5, 0.8)^\top$ calculated using the third tubes. At the bottom of Figure 3, we once again present the trajectories and also several cross-sections $\mathcal{P}^-[k]$ of the used polyhedral solvability tubes $\mathcal{P}^-[\cdot]$ to show the dynamics of cross-sections. We see that $\mathcal{P}^-[k]$ satisfy the state constraints. The controlled trajectories obtained reach the target set and also satisfy the state constraints.

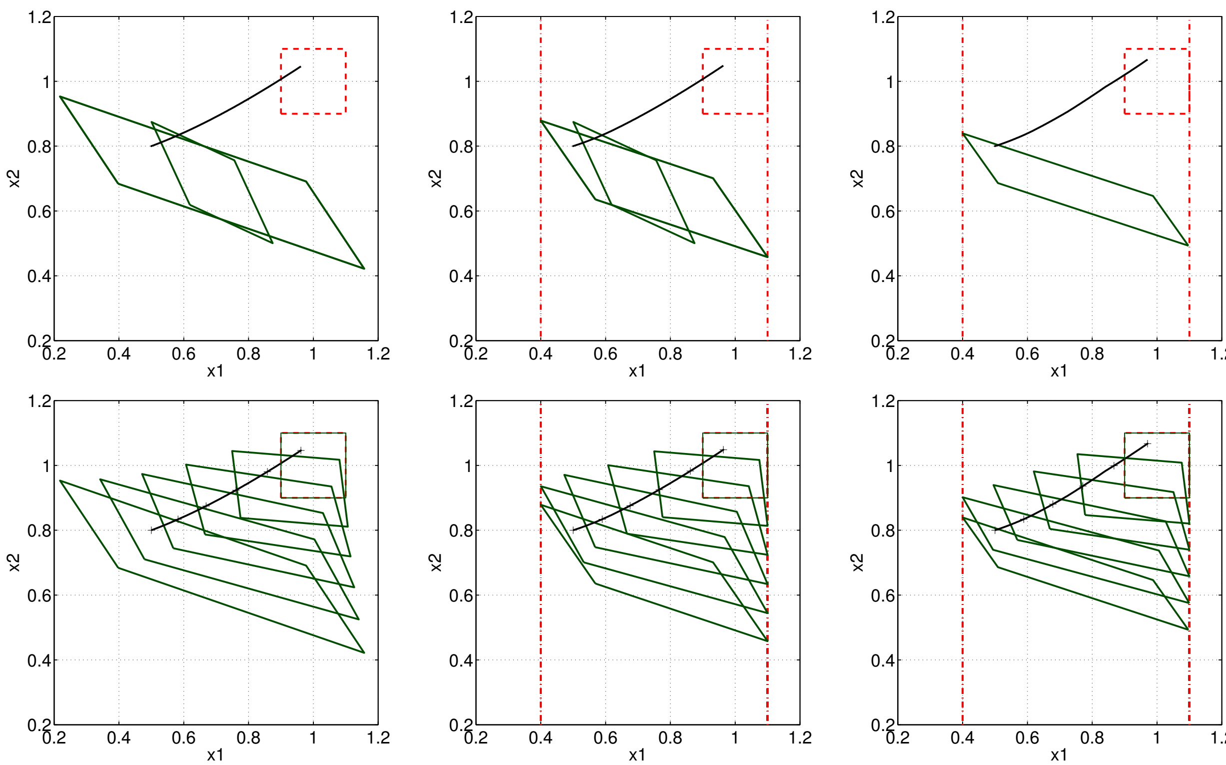

Fig. 3. Results of polyhedral synthesis in Example 5.2 for cases (A), (A;SC), and (B,ii;SC)

Conclusion

Two types of problems of feedback terminal target control for linear and bilinear discrete-time uncertain systems under state constraints are considered, where controls appear either additively or in the system matrix. Polyhedral control synthesis using polyhedral (parallelotope-valued) solvability tubes is elaborated. The cases without uncertainties, with additive uncertainties, and also with a matrix uncertainty are considered. Nonlinear recurrent relations are presented for polyhedral solvability tubes. Control strategies, which can be calculated by explicit formulas on the base of these tubes, are proposed. Proposed polyhedral solvability tubes may turn out to be rather conservative. But we can easily compute them, while the maximal solvability tubes are hard to construct. One should also bear in mind that the use of the strategies proposed for Problem 3 does not exhaust, in general, all possibilities to control systems (1), (3), (4)–(6) with controls in matrix.

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