GENERALIZED SEMI-INFINITE OPTIMIZATION: ON SOME FOUNDATIONS

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Рассматривается задача полубесконечной оптимизации в общем виде

$$\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v) \qquad \begin{cases} \text{Min} \quad f(x) \ \text{ на } \ M_{\mathcal{SI}}[h,g], \quad \text{где} \\ \\ M_{\mathcal{SI}}[h,g] \ := \ \{\, x \in I\!R^n \mid \ h_i(x) = 0 \ \ (i \in I), \\ \qquad\qquad\qquad\qquad g(x,y) \geq 0 \ \ (y \in Y(x))\,\}, \end{cases}$$

в которой $Y(x) = M_{\mathcal{F}}[u(x,\cdot), v(x,\cdot)]$ $(x \in I\!R^n)$ — допустимые множества в смысле оптимизации с конечными ограничениями. Предполагается, что некоторые ограничения выполнены для $Y(x)$, (LICQ или MFCQ), локально (или глобально) по x. Задача $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$ может быть локально (или глобально) представлена как обычная полубесконечная оптимизационная задача. Таким образом, используются два различных подхода, каждый из которых с или без предположения компактности на $Y(x)$. Более того, для $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$ мы предлагаем необходимые и, при некоторых дополнительных предположениях, достаточные условия оптимальности первого порядка, которые в специальном случае были впервые предложены Кайзером и Крабсом.

1. Introduction, a first equivalent model

In semi-infinite optimization very often the index set Y of maybe infinitely many inequality constraints does not depend on the state $x \in I\!R^n$. The set of equality constraints is finite. For problems of this kind which we call ordinary semi-infinite optimization problems, a lot of research has been done; we mention, e. g., [4, 13, 26] and, from special viewpoints, [21, 23, 24]. In this research, however, we are concerned with generalized semi-infinite optimization problems. These problems have the form

$$\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v) \qquad \begin{cases} \text{Minimize} \quad f(x) \ \text{ on } \ M_{\mathcal{SI}}[h,g], \quad \text{where} \\ \\ M_{\mathcal{SI}}[h,g] \ := \ \{\, x \in I\!R^n \mid \ h_i(x) = 0 \ \ (i \in I), \\ \qquad\qquad\qquad\qquad g(x,y) \geq 0 \ \ (y \in Y(x))\,\} \\ \\ \text{and} \\ \qquad Y(x) \ := \ M_{\mathcal{F}}[u(x,\cdot), v(x,\cdot)] \ \ (x \in I\!R^n), \end{cases}$$

i. e., now the index set Y, with its perhaps infinitely many elements, depends on the state $x \in \mathbb{R}^n$. Moreover, $Y(x)$ is a feasible set in the sense of finitely constrained optimization ($\mathcal{F}$), i. e., the set of inequality constraints has only finitely many elements. For our problem, $\mathcal{SI}$ abbreviates its semi-infiniteness.

Let h, u, v comprise the component functions $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I := \{1,\dots,m\}$, $u_k : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $k \in K := \{1,\dots,r\}$, and $v_\ell : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $\ell \in L := \{1,\dots,s\}$, respectively. Moreover, we assume that $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $h_i\,(i \in I)$, $u_k\,(k \in K)$ and $v_\ell\,(\ell \in L)$ are C^1-functions (continuously differentiable). For each C^1-function, e. g. for f, $Df(x)$ and $D^T f(x)$, denote the row- and the column-vector of the first order partial derivatives $\dfrac{\partial}{\partial x_\kappa} f(x)$ $(\kappa \in \{1,\dots,n\};\ x \in \mathbb{R}^n)$, respectively. Let, e. g., $D_x g(x,y)$, $D_y g(x,y)$, analogously comprise the partial derivatives of g due to the coordinates x_κ and y_σ, respectively (and so on for the other defining functions).

For problems of that generalized form we refer, from the topological (generic) point of view, to [12, 20]. Hereby, a basic aspect is the question whether the so-called Reduction-Ansatz applies or, to what an extent it is violated. That Reduction-Ansatz means the opportunity locally (e. g., around certain critical points) to express the semi-infinite optimization problem as a finite optimization problem (cf. [11, 33]). This paper, however, is based on the question of local (or global) representability of our generalized semi-infinite optimization problem as an ordinary semi-infinite optimization problem. Hereby, we walk along two different ways, the first of which is presented in Section 1, the other one in Section 2. The two ways (approaches, models) are (critically) commented and compared.

In both of these ways, we assume for $Y(x) \subseteq \mathbb{R}^q$, locally in x, the linear independence constraint qualification, the Mangasarian — Fromovitz constraint qualification, or a connectedness property. Finally, we shall refer to a local minimum $\hat{x}$ and to a small neighborhood of this point. Then, approaching our models with the help of appropriate other assumptions, we are in a position to derive first order necessary or sufficient optimality conditions, and, in the further paper [32], the iteration procedures which were presented by Kaiser and Krabs [28]. In [28] the authors in particular refer to the special case of an interval $Y(x) := [a(x), b(x)]$ $(q = 1)$. Kaiser's and Krabs' investigation was motivated by a concrete problem from mechanical engineering (see [2]). That problem consists in

— *optimizing the layout of a special assembly line:*

under certain constraints, the objects to be transported in periodic time intervals, shall be moved as far as possible in each period. (For the modelling see also [27].)

As some further examples which can, under appropriate assumptions, be modelled as a generalized semi-infinite optimization problem we mention the following problems:

— *reverse Chebychev approximation*:

motivated by the approximation of a thermocouple characteristic in chemical engineering [34] or by digital filtering (cf. [15, 19, 25]),

— *maneuverability of a robot* (cf. [7, 19, 25]),

— *time optimal control* (e. g., time minimal heating or cooling of a ball of some homogeneous material; cf. [25]),

— *structure and stability in optimal control of an ordinary differential equation* (cf. [31]).

We remark that in [28] there are also considerations on sets $Y(x)$ which do not have the structure of a feasible set.

This present research has also some relations with the recent article [19] of Jongen, Rückmann and Stein. While in [19] as the main result a general Fritz — John type necessary optimality theorem is proved, here our necessary optimality conditions are closer to the Kuhn—Tucker type. Mainly, however, we shall for a local minimum $\hat{x}$ describe the necessary nonnegativity of the function $\xi \mapsto Df(\hat{x})\xi$ $(\xi \in I\!R^n)$ over some linearized tangent cone of the feasible set at $\hat{x}$. Finally, in [19] a Kuhn—Tucker theorem is concluded, too. Moreover, in [19] assumptions on compactness of $Y(x)$ $(x \in I\!R^n)$ and on upper semi-continuity of $Y(\cdot)$ in the sense of Berge are made, while no constraint qualification is assumed. In our paper such a compactness (or boundedness) assumption is discussed, but not always necessary, and a (pointwise) constraint qualification or some connectedness property with respect to $Y(\cdot)$ is assumed. (The article [19] and the present research with their results, proofs and further considerations were started independently, under different aspects and motivating examples.)

These problem representations and optimality conditions play an important part for creating methods to find local or global minima. In this context of iteration procedures, topological questions of the behavior of the feasible set $M_{\mathcal{SI}}[h,g]$ under data perturbations naturally arise. In [32] these questions are treated by means of generalizing some of the results on ordinary semi-infinite optimization made by Jongen, Twilt and Weber ([21]).

We introduce the following two boundedness assumptions:

Assumption A (Boundedness). *There is a bounded open neighborhood* $\mathcal{U} \subset I\!R^n$ *of* $M_{\mathcal{SI}}[h,g](\subset \mathcal{U})$, *such that* $\cup_{x\in\overline{\mathcal{U}}} Y(x)$ *is bounded*,
and, as a weaker condition,

Assumption $\mathrm{A}_{\mathcal{U}^0}$ (Boundedness, locally). *There is a bounded set* $\mathcal{U}^0 \subset I\!R^n$ *with* $M_{\mathcal{SI}}[h,g] \cap \mathcal{U}^0 \neq \emptyset$ *such that* $\cup_{x\in\overline{\mathcal{U}^0}} Y(x)$ *is bounded.*

Hereby, e. g., $\overline{\mathcal{U}}$ denotes the closure of $\mathcal{U}$. In the sequel we shall precisely discuss, to what an extent global or local assumptions on compactness, or constraint qualifications, should be satisfied.

Under these differentiability and, hence, continuity assumptions all the sets $Y(x)$ $(x \in I\!R^n)$ are closed. By means of a small argumentation on continuity and compactness, we see that the bounded sets $\cup_{x\in\overline{\mathcal{U}}}Y(x), \cup_{x\in\overline{\mathcal{U}^0}}Y(x)$ from Assumptions A and $\mathrm{A}_{\mathcal{U}^0}$, respectively, are even compact. Hence, from now on we may call the Assumptions A and $\mathrm{A}_{\mathcal{U}^0}$, *compactness assumptions.*

Remark[1]. In the case of any constraint qualification on $Y(x)$ (for all $x \in \overline{\mathcal{U}}$ or for all $x \in \overline{\mathcal{U}^0}$) which we shall make, and if due to no converging sequence $(x^\kappa)_{\kappa\in I\!N}$ (in $\overline{\mathcal{U}}$ or $\overline{\mathcal{U}^0}$, respectively) and for no sequence $(y^\kappa)_{\kappa\in I\!N}$ lying "at infinity", the sequences $(u_k(x^\kappa, y^\kappa))_{\kappa\in I\!N}$ $(k \in K)$, and $(\min_{\ell\in L} v_\ell(x^\kappa, y^\kappa))_{\kappa\in I\!N}$, approach 0 or their members are non negative, respectively, then that compactness (of the unions) is not only sufficient but also necessary for the boundedness (compactness) of each of these sets $Y(x)$. This fact follows from a topological consideration based on [21].

For the set of active inequality constraints at a elements $\overline{x}^1 \in M_{\mathcal{SI}}[h,g]$, $\overline{x}^2 \in I\!R^n$ and at an element $\overline{y} \in M_{\mathcal{F}}[u(\overline{x}^2,\cdot), v(\overline{x}^2,\cdot)]$ (or, lateron, $\overline{y} \in I\!R^q$), respectively, we write

$$Y_0(\overline{x}^1) := \{y \in Y(\overline{x}^1) | g(\overline{x}^1, y) = 0\}, \tag{1.1}$$

$$L_0(\overline{x}^2, \overline{y}) := \{\ell \in L | v_\ell(\overline{x}^2, \overline{y}) = 0\}. \tag{1.2}$$

Definition 1.1. Let points $\overline{x} \in I\!R^n$ and $\overline{y} \in Y(\overline{x})$ be given. We say that the *linear independence constraint qualification*, in short: *LICQ*, holds at $\overline{y}$ as an element of the feasible set $M_{\mathcal{F}}[u(\overline{x},\cdot), v(\overline{x},\cdot)]$ if the vectors

$$D_y u_k(\overline{x}, \overline{y}) \quad (k \in K), \quad D_y v_\ell(\overline{x}, \overline{y}) \quad (\ell \in L_0(\overline{x}, \overline{y}))$$

[1]If the reader is not so much interested in additional theoretical details, then he may skip the remark.

are linearly independent.

The *linear independence constraint qualification (LICQ)* is said to hold for $M_{\mathcal{F}}[u(\overline{x},\cdot),v(\overline{x},\cdot)]$ if LICQ is fulfilled for all $y \in Y(\overline{x})$.

Assumption B (LICQ). *LICQ holds for all sets $M_{\mathcal{F}}[u(x,\cdot),v(x,\cdot)]$ $(x \in \overline{\mathcal{U}})$ where $\mathcal{U} \subseteq I\!R^n$ is an open neighborhood of $M_{\mathcal{SI}}[h,g]$ (cf., e. g., Assumption A),*
or

Assumption $\mathrm{B}_{\mathcal{U}^0}$ (LICQ, locally). *Referring to some given open set $\mathcal{U}^0 \subseteq I\!R^n$ (cf., e. g., Assumption $\mathrm{A}_{\mathcal{U}^0}$) LICQ holds for all sets $M_{\mathcal{F}}[u(x,\cdot),v(x,\cdot)](x \in \overline{\mathcal{U}^0})$.*

The topological investigation [18] on parametric finitely constrained optimization indicates that the Assumption B on (with the parameter x) global validity of LICQ is very strong (cf. also [10, 22]). As for our purposes of optimality conditions we do not globally need to represent our generalized semi-infinite problem by an ordinary one, but locally, Assumption $\mathrm{B}_{\mathcal{U}^0}$ will finally be sufficient. Hereby, on the one hand the special case $\mathcal{U}^0 = \mathcal{U}$ will be included in our general considerations. On the other hand, due to a small neighborhood $\mathcal{U}^1$ of a given point $\hat{x} \in I\!R^n$, the validity for $Y(\hat{x})$ of both LICQ and Assumption $\mathrm{A}_{\mathcal{U}^1}$ is sufficient for Assumption $\mathrm{B}_{\mathcal{U}^0}$ to hold, where $\mathcal{U}^0 \subseteq \mathcal{U}^1$ is a sufficiently small other neighborhood of $\hat{x}$.

Only for both the ease of the following (re)presentation of our problem with the help of an ordinary semi-infinite optimization problem and in order to work out the case of locally holding compactness being important for iteration procedures, we may without loss of generality make the Assumption $\mathrm{A}_{\mathcal{U}^0}$.

Nevertheless, until we arrive at that ordinary semi-infinite problem $\mathcal{P}^o_{\mathcal{SI}}(f,h,g^0,u^0,v^0)$, our explanations need some technical effort.[2]

Then, Assumption $\mathrm{B}_{\mathcal{U}^0}$ gives us the opportunity, for a fixed $\overline{x} \in \overline{\mathcal{U}^0}$ locally around a given point $\overline{y} \in Y(\overline{x})$ to linearize $Y(x)$ for x close to $\overline{x}$ (cf. [16, 22]). Therefore we define $z = \widehat{F}(x,y)$ as follows:

$$\left.\begin{array}{rcl} z_1 &:=& u_1(x,y) \\ &\vdots& \\ z_r &:=& u_r(x,y) \\ z_{r+1} &:=& v_{\ell^1}(x,y) \\ &\vdots& \\ z_{r+p} &:=& v_{\ell^p}(x,y) \\ z_{r+p+1} &:=& \xi_1^T(y-\overline{y}) \\ &\vdots& \\ z_q &:=& \xi^T_{q-r-p}(y-\overline{y}) \end{array}\right\} \quad ((x,y) \in I\!R^n \times I\!R^q), \tag{1.3a}$$

where p is the cardinality $|L_0(\overline{x},\overline{y})|$ of $L_0(\overline{x},\overline{y}) = \{\ell^1,\ldots,\ell^p\}$ and where the vectors $\xi_\eta \in I\!R^q$ $(\eta \in \{1,\ldots,q-r-p\})$ complete the set $\{\, D_y u_k(\overline{x},\overline{y}) \mid k \in K \,\} \cup \{\, D_y v_\ell(\overline{x},\overline{y}) \mid \ell \in L_0(\overline{x},\overline{y}) \,\}$ to a basis of $I\!R^q$.

Let us put

$$\widehat{\mathcal{F}}(x,y) := (x, \widehat{F}(x,y)) \quad ((x,y) \in I\!R^n \times I\!R^q). \tag{1.3b}$$

[2] If the reader is not very interested in the topological details, he might skip those details and immediately turn to Theorem 1.2.

Then, we realize that the partitioned matrix of derivatives of the component functions

$$D\widehat{\mathcal{F}}(\overline{x},\overline{y}) = \left(\begin{array}{c|c} I_n & O_{n\times q} \\ \hline D_x\widehat{F}(\overline{x},\overline{y}) & D_y\widehat{F}(\overline{x},\overline{y}) \end{array}\right), \tag{1.4a}$$

where I_n = unit-matrix of $I\!R^n$ and $O_{n\times q}$ = zero-matrix of type $n \times q$, is nonsingular if and only if the matrix

$$D_y\widehat{F}(\overline{x},\overline{y}) = \begin{pmatrix} \vdots \\ D_y u_k(\overline{x},\overline{y}) \ \ (k \in K) \\ \vdots \\ D_y v_\ell(\overline{x},\overline{y}) \ \ (\ell \in L_0(\overline{x},\overline{y})) \\ \vdots \\ \xi_\eta^T \ \ (\eta \in \{1,\ldots,q-r-p\}) \\ \vdots \end{pmatrix} \tag{1.4b}$$

is nonsingular. The latter condition, however, is guaranteed by Assumption $\mathrm{B}_{\mathcal{U}^0}$ and by the choice of the vectors ξ_η. Now, by means of applying the inverse function theorem at $(\overline{x},\overline{y})$ on $\widehat{\mathcal{F}}$ we conclude that there exist open and bounded neighborhoods $\mathcal{U}_1 \subseteq I\!R^n$, $\mathcal{U}_2 \subseteq I\!R^q$ around $(\overline{x},\ \overline{y})$ respectively, such that $\mathcal{F} := \widehat{\mathcal{F}}|(\mathcal{U}_1 \times \mathcal{U}_2) : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{W} := \widehat{\mathcal{F}}(\mathcal{U}_1 \times \mathcal{U}_2)$ is a C^1-diffeomorphism. Now, let $||\cdot||_\infty$ denote the maximum norm of the underlying Euclidean space. Shrinking $\mathcal{U}_1, \mathcal{U}_2$, if necessary, we can guarantee that on the one hand $\mathcal{W}$ is an axis-parallel open cube around $(\overline{x}, 0_q) \in I\!R^n \times I\!R^q$. This means $\mathcal{W} = \mathcal{C}_1 \times \mathcal{C}_2$, where $\mathcal{C}_1 = C(\overline{x}, \rho^1) := \{\, x \in I\!R^n \mid ||x-\overline{x}||_\infty < \rho^1 \,\}$, $\mathcal{C}_2 = C(0_p, \rho^2)$ stand for the open cubes $\mathcal{C}_1$, $\mathcal{C}_2$ of the $||\cdot||_\infty$-radii (= half the length) ρ^1, ρ^2, around $\overline{x}$, 0_p, respectively, and being parallel with respect to the axis. On the other hand, we have $L_0(x,y) \subseteq L_0(\overline{x},\overline{y})$ for all $(x,y) \subseteq \mathcal{U}_1 \times \mathcal{U}_2$. Then, for each $x \in \mathcal{U}_1$ the mapping $\phi_x := (\widehat{F}(x,\cdot))|\mathcal{U}_2 : \ \mathcal{U}_2 \longrightarrow \mathcal{C}_2$ is a C^1-diffeomorphism which also transforms the (relative) neighborhood $Y(x) \cap \mathcal{U}_2$ of $\overline{y}$ onto the (relative) neighborhood

$$(\{0_r\} \times I\!H^p \times I\!R^{q-r-p}) \cap \mathcal{C}_2 \ \subseteq \ I\!R^q$$

of 0_q. Hereby, $I\!H^p$ denotes the nonnegative orthant $\{\, z \in I\!R^p \,|\, z_\ell \geq 0 \ (\ell \in \{1,\ldots,p\}) \,\}$ of $I\!R^p$. We call ϕ_x a *canonical local change of the coordinates* (of) y.

For all points $x \in \mathcal{U}_1$, $z \in \mathcal{C}_2$ we have the pre-image for $\widehat{F}(x,\cdot)$, $\widehat{F}(x,\cdot) - z$ of the corner point 0_q pointwise being given by means of *implicit* C^1*-functions* $\underline{y}(\cdot)$, $\hat{y}(\cdot,\cdot)$ of x and (x,z), respectively, i. e.

$$\phi_x^{-1}(0_q) = \underline{y}(x), \quad |L_0(x,\underline{y}(x))| \equiv p \quad (x \in \mathcal{U}_1), \tag{1.5a}$$

$$\phi_x^{-1}(z) = \hat{y}(x,z). \tag{1.5b}$$

Performing the construction of a C^1-family $(\phi_x^{\overline{a}})_{x \in \mathcal{U}_1^{\overline{a}}}$ of diffeomorphisms $\phi_x^{\overline{a}} : \mathcal{U}_2^{\overline{a}} \to \mathcal{C}_2^{\overline{a}}$ for each $\overline{a} = (\overline{x},\overline{y})$ where $\overline{x} \in \overline{\mathcal{U}^0}$, $\overline{y} \in Y(\overline{x})$. In particular, we may choose an open covering $(\widetilde{\mathcal{U}}_1^{\overline{a}} \times \widetilde{\mathcal{U}}_2^{\overline{a}})_{\overline{a}\in\mathcal{A}}$ of $\mathcal{A} := \{(\hat{x},\hat{y}) \,|\, \hat{x} \in \overline{\mathcal{U}^0},\ \hat{y} \in Y(\hat{x})\}$ where $\overline{x} \in \widetilde{\mathcal{U}}_1^{\overline{a}} \subset \overline{\widetilde{\mathcal{U}}_1^{\overline{a}}} \subset \mathcal{U}_1^{\overline{a}}$, $\overline{y} \in \widetilde{\mathcal{U}}_2^{\overline{a}} \subset \overline{\widetilde{\mathcal{U}}_2^{\overline{a}}} \subset \mathcal{U}_2^{\overline{a}}$, and where $\widetilde{\mathcal{W}}^{\overline{a}} := \widehat{\mathcal{F}}(\widetilde{\mathcal{U}}_1^{\overline{a}} \times \widetilde{\mathcal{U}}_2^{\overline{a}})$ is an (axis-parallel) open subcube $\widetilde{\mathcal{W}}^{\overline{a}} = \widetilde{\mathcal{C}}_1^{\overline{a}} \times \widetilde{\mathcal{C}}_2^{\overline{a}}$ of $\mathcal{W} = \mathcal{W}^{\overline{a}}$.

From Assumption $A_{\mathcal{U}^0}$, by means of an argumentation with (sub)sequences we conclude that $\mathcal{A}$ is compact. Hence, there exist finitely many points $\overline{a}^j = (\overline{x}^j, \overline{y}^j) \in \mathcal{A}$, $j \in J := \{1, \ldots, w\}$, such that $(\widetilde{\mathcal{U}}_1^{\overline{a}^j} \times \widetilde{\mathcal{U}}_2^{\overline{a}^j})_{j \in J}$ is an open covering of $\mathcal{A}$.

We note that in the general case without a compactness assumption, this subcovering can be chosen with $J \subseteq I\!N$ and locally finite. This means: for each $\tilde{a} := (\tilde{x}, \tilde{y}) \in \{(x, y) \in \overline{\mathcal{U}^0} \times I\!R^q \,|\, y \in Y(x)\}$ there is a neighbourhood $\mathcal{V}'^{\tilde{a}}$ such that $\mathcal{V}'^{\tilde{a}} \cap (\widetilde{\mathcal{U}}_1^{\overline{a}^j} \times \widetilde{\mathcal{U}}_2^{\overline{a}^j}) \neq \emptyset$ for only finitely many $j \in J$. (Hereby, the open set $\mathcal{U}^0$ needs not to be bounded.)

Let us give the idea how to achieve such a locally finite structure. Therefore, we can decompose the set $\mathcal{A}$ by means of intersecting it with the countably many compact cubes $\mathsf{C}^\nu = \mathsf{C}_1^\nu \times \mathsf{C}_2^\nu := [2\nu, 2\nu+1]^{n+q} \subset I\!R^n \times I\!R^q$ ($\nu \in Z\!\!\!Z$, i.e., ν is an integer). These cubes altogether cover $I\!R^n \times I\!R^q$. Let some $\nu_0 \in Z\!\!\!Z$ be given. The intersection $\mathcal{A} \cap \mathsf{C}^{\nu_0} = \{(\hat{x}, \hat{y}) \mid \hat{x} \in \overline{\mathcal{U}^0} \cap \mathsf{C}_1^{\nu_0}$, $\hat{y} \in Y(\hat{x}) \cap \mathsf{C}_2^{\nu_0}\}$ is compact. Hence, in C^{ν_0} we are in a similar situation as under Assumption $A_{\mathcal{U}^0}$, such that we may choose a finite open covering $\mathcal{O}^{\nu_0}$ of $\mathcal{A} \cap \mathsf{C}^{\nu_0}$. Taking into account all these open coverings $\mathcal{O}^\nu$ ($\nu \in Z\!\!\!Z$) and enumerating all their members (open sets) by means of $j \in J \subseteq I\!N$, we finally arrive at a suitable locally finite open covering of $\mathcal{A}$. If $\mathcal{A}$ is actually known not to be compact (the case of unboundedness), then we may even choose $J = I\!N$.

Now, with our (local) linearizations of $Y(x)$ ($x \in \mathcal{U}_1^{\overline{a}}$, $\overline{a} \in \mathcal{A}$) we are able equivalently to represent our inequality constraints on x (on the "upper stage") without x-dependence of the index set. In fact, writing $p^j := |L_0(\overline{x}^j, \overline{y}^j)|$, $Z^j := \overline{(\{0_r\} \times I\!H^{p^j} \times I\!R^{q-r-p^j}) \cap \widetilde{\mathcal{C}}_2^{\overline{a}^j}}$ ($j \in \{1, \ldots, w\}$) and ϕ_x^j, $\mathcal{C}_2^j$, $\widetilde{\mathcal{C}}_2^j$ for $\phi_x^{\overline{a}^j}$, $\mathcal{C}_2^{\overline{a}^j}$, $\widetilde{\mathcal{C}}_2^{\overline{a}^j}$, we have for each $x \in \overline{\mathcal{U}^0}$:

$$\left.\begin{array}{rl} & g(x,y) \geq 0 \quad \text{for all } y \in Y(x) \\[2ex] \Longleftrightarrow & g(x, (\phi_x^j)^{-1}(z)) \geq 0 \ \text{ for all } z \in (\{0_r\} \times I\!H^{p^j} \times I\!R^{q-r-p^j}) \cap \mathcal{C}_2^j, \\[1ex] & \hfill \text{if } x \in \mathcal{U}_1^{\overline{a}^j},\ j \in J; \\[2ex] \Longleftrightarrow & g(x, (\phi_x^j)^{-1}(z)) \geq 0 \quad \text{for all } z \in Z^j, \qquad \text{if } x \in \widetilde{\mathcal{U}}_1^{\overline{a}^j},\ j \in J. \end{array}\right\} \tag{1.6}$$

Finally, let us for each $\kappa \in \{1, \ldots, \kappa^0\}$ by means of our set inclusions from above and of a C^∞- partition of unity (cf. [14, 17]) glue together $(x, z) \mapsto g(x, (\phi_x^j)^{-1}(z))$ with 0 in $\mathcal{V}^j := \mathcal{U}_1^{\overline{a}^j} \times \mathcal{C}_2^j$. Namely, we let the resulting function g_j^0 coincide with $g(x, (\phi_x^j)^{-1}(z))$ for all $(x, z) \in \overline{\widehat{\mathcal{V}}_1^j}$ and with 0 on $\mathcal{V}^j \setminus \widehat{\mathcal{V}}_2^j$. Hereby, $\widehat{\mathcal{V}}_1^j$, and $\widehat{\mathcal{V}}_2^j$ are open subsets of $\mathcal{V}^j$ being chosen such that with $\widetilde{\mathcal{V}}^j := \widetilde{\mathcal{U}}_1^{\overline{a}^j} \times \widetilde{\mathcal{C}}_2^j$ it holds $\overline{\widetilde{\mathcal{V}}^j} \subset \widehat{\mathcal{V}}_1^j$, $\overline{\widehat{\mathcal{V}}_1^j} \subset \widehat{\mathcal{V}}_2^j$, $\overline{\widehat{\mathcal{V}}_2^j} \subset \mathcal{V}^j$. So, we have immediately extended those functions g_j^0 by means of 0 outside of $\widehat{\mathcal{V}}_1^j$ ($j \in J$; note that $\hat{y}$ in (1.5b) is C^1). As each of our gluing partitions of unity is an ((x, z)-dependent) convex combinations of values which are lower bounded by 0, it has the same property, too. Hence, we may for each $x \in \overline{\mathcal{U}^0}$ conclude from (1.6):

$$g(x, y) \geq 0 \quad \text{for all } y \in Y(x) \qquad \Longleftrightarrow \qquad g_j^0(x, z) \geq 0 \quad \text{for all } z \in Z^j,\ j \in J. \tag{1.7}$$

We note that, by definition, each of the new index sets Z^j is a $(q - r)$-dimensional closed cube with 0_q being one of its corner points. In particular, the sets Z^j are feasible sets in the sense of finite optimization,

$$Z^j \ = \ M_{\mathcal{F}}[u_j^0, v_j^0] \ = \ \{0_r\} \times [0, b_1^j] \times \cdots \times [0, b_{q-r}^j] \tag{1.8}$$

where $u_j^0 = (u_{j\,1}^0, \ldots, u_{j\,r}^0)$, $u_{j\,k}^0(z) := z_k$ $(z \in I\!R^q,\ k \in K = \{1, \ldots, r\},\ j \in J)$ and where the components $v_{j\,\ell}^0$ of $v_j^0 = (v_{j\,1}^0, \ldots, v_{j\,2(q-r)}^0)$ reflect the boundary points of $q - r$ coordinate-wise intervals: $v_{j\,2\ell-1}^0(z) = z_\ell \geq 0$, $v_{j\,2\ell}^0(z) = -z_\ell \geq -b_\ell^j$ $(\ell \in \{1, \ldots, q-r\})$. Moreover, because of their forms these feasible sets Z^j are compact and fulfill the condition *LICQ*. Let us shortly write $g^0 := (g_1^0, \ldots, g_w^0)$. Now, we finally arrived at the problem

$$\mathcal{P}^o_{\mathcal{SI}}(f, h, g^0, u^0, v^0) \qquad \begin{cases} \text{Minimize} \quad f(x) \ \text{ on } \ M^o_{\mathcal{SI}}[h, g^0], \quad \text{where} \\[2ex] M^o_{\mathcal{SI}}[h, g^0] \ := \ \{\, x \in I\!R^n \mid \ h_i(x) = 0 \ \ (i \in I), \\[2ex] \qquad\qquad\qquad g_j^0(x, z) \geq 0 \ \ (z \in Z^j,\ j \in J) \,\}. \end{cases}$$

We remember that in our *general case* where only the Assumption $\mathrm{B}_{\mathcal{U}^0}$ is made, $J \subseteq I\!N$ needs not to be finite. Hence, g^0 may be a sequence, e. g., $g^0 = (g_j^0)_{j \in I\!N}$.

Theorem 1.2. *Let the Assumption $B_{\mathcal{U}^0}$ on LICQ hold, due to a given open set $\mathcal{U}^0 \subseteq I\!R^n$, $M_{\mathcal{SI}}[h, g] \cap \mathcal{U}^0 \neq \emptyset$, and for the given generalized semi-infinite optimization problem $\mathcal{P}_{\mathcal{SI}}(f, h, g, u, v)$. We assume that the defining functions of this problem are of class C^1.*

Then, in $\overline{\mathcal{U}^0}$, $\mathcal{P}_{\mathcal{SI}}(f, h, g, u, v)$ can always equivalently be expressed as an ordinary semi-infinite optimization problem $\mathcal{P}^o_{\mathcal{SI}}(f, h, g^0, u^0, v^0)$ with defining functions of class C^1, too.

Moreover, for its feasible set $M^o_{\mathcal{SI}}[h, g^0]$ we have

$$M^o_{\mathcal{SI}}[h, g^0] \cap \overline{\mathcal{U}^0} \ = \ M_{\mathcal{SI}}[h, g] \cap \overline{\mathcal{U}^0}, \tag{1.9a}$$

where the sets $Z^j = M_{\mathcal{F}}[u_j^0, v_j^0]$ $(j \in J)$ of inequality constraints are compact and fulfill LICQ.

If also Assumption $A_{\mathcal{U}^0}$ holds, then J is finite and in the special case $\mathcal{U}^0 = \mathcal{U}$ (Assumption A) (1.9a) means

$$M^o_{\mathcal{SI}}[h, g^0] = M_{\mathcal{SI}}[h, g]. \tag{1.9b}$$

Note that in the special case $\mathcal{U}^0 = \mathcal{U}$ the equation (1.9b) follows from both (1.9a) and the inclusions $M_{\mathcal{SI}}[h, g] \subset \mathcal{U}$, $M^o_{\mathcal{SI}}[h, g^0] \subset \mathcal{U}$. Hereby the last inclusion is guaranteed by the construction of g^0. We emphasize that in this case of Assumption A our Theorem 1.2 gives us a locally equivalent formulation of $\mathcal{P}_{\mathcal{SI}}(f, h, g, u, v)$ as an ordinary semi-infinite problem.

In analogy with (1.1), (1.2) we introduce for each $\overline{x} \in M^o_{\mathcal{SI}}[h, g^0]$ the following "active sets":

$$Z_0^j(\overline{x}) \ := \ \{\, z \in Z^j \ \mid \ g_j^0(\overline{x}, z) \ = \ 0 \,\} \qquad (j \in J), \tag{1.10}$$

$$Z_0^o(\overline{x}) \ := \ \{\, (j, z) \in J \times I\!R^q \ \mid \ z \in Z_0^j(\overline{x}) \,\}. \tag{1.11}$$

Remark. We note that there is some ambiguity in the activity behaviour between our generalized semi-infinite problem and the ordinary semi-infinite problem. Namely, because of $\mathcal{P}^o_{\mathcal{SI}}(f, h, g^0, u^0, v^0)$ being introduced by means of open coverings there may be a point $x \in M_{\mathcal{SI}}[h, g]$ with an active index $y \in Y_0(x)$ corresponding to (more than one) different indices $(j^\kappa, z^{j^\kappa}) \in Z_0^o(x)$ $(\kappa \in \{1, \ldots, \kappa'\},\ \kappa' \in I\!N,\ \kappa' > 1,\ |\{j^1, \ldots, j^{\kappa'}\}| = \kappa')$.

Finally, in the context of Kuhn — Tucker conditions we have to face a (special) disadvantage which comes from the definition of Z^j $(j \in J)$. Namely, for the definition of theses sets further inequalities are involved which do not represent one of the inequalities v_ℓ $(\ell \in L)$. This should give rise to care for some fineness of our open coverings.

In our second approach on expressing $\mathcal{P}_{\mathcal{SI}}(f, h, g, v)$ $(K = \emptyset)$ as an ordinary semi-infinite problem in Section 2, that ambiguity and this disadvantage do not exist. Moreover, the second approach does not need the formalism of changing the coordinates (diffeomorphisms, inversions).

Let us make a last technical preparation for the next section. Therefore, we assume $\hat{x} \in \mathcal{U}^0$ to be a fixed feasible, maybe a local minimal point for $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$. Then, $\overline{a}^j = (\overline{x}^j, \overline{y}^j)$ $(j \in J)$ can always be chosen such that $\hat{x} = \overline{x}^j$ whenever $\hat{x} \in \widetilde{\mathcal{U}}_1^{\overline{a}^j}$ for some $j \in J$.

Before we turn to optimality conditions for $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$, i. e. for $\mathcal{P}^o_{\mathcal{SI}}(f,h,g^0,u^0,v^0)$ in $\overline{\mathcal{U}^0}$, we make two remarks on the new formulation of our problem.

Remarks. (a) (An analytical remark.) Besides the critical comment from the remark above and in comparison with the (maybe infinitely many) index sets $Y(x)$ $(x \in \overline{\mathcal{U}^0})$ the (locally in x, finitely many) index sets Z^j $(j \in J)$ have the further advantage of being linearized. These sets were introduced more implicitly (inverse or implicit function theorem); however, there is some information on their sizes.

Indeed, for each of the sets Z^j $(j \in J)$ we have a "controlling" parameter $\beta^j > b^j_\ell$ $(\ell \in \{1,\ldots,q-r\})$ in order to estimate in the maximum norm $||\cdot||_\infty$ the (coordinate-wise defined) size $\max\{\, b^j_\ell \,|\, \ell \in \{1,\ldots,q-r\}\}$ of Z^j. Therefore, we consider the proof of the inverse function theorem which is based on a suitable application of Banach's fixed point theorem (see, e. g., [1]). Then, we see in view of (1.3, a, b) that β^j should for each $(x,y) \in I\!R^n \times I\!R^q$ with $||(x,y) - (\overline{x}^j, \overline{y}^j)||_\infty \leq 2\beta^j$ satisfy $||I_{n+q} - D\widehat{\mathcal{F}}(x,y)||_\infty \leq \dfrac{1}{2}$. In detail, the last inequality means

$$\left\| \left(\frac{A(x,y)}{O_{(q-r-p^j)\times n}} \;\middle|\; \left(\frac{B(x,y)}{C}\right) - I_q \right) \right\|_\infty \leq \frac{1}{2},$$

where $A = (\ldots,\, D_x^T u_k\ (k \in K),\, \ldots,\, D_x^T v_\ell\ (\ell \in L_0(\overline{x}^j, \overline{y}^j)),\, \ldots)^T$, $B = (\ldots,\, D_y^T u_k\ (k \in K),\, \ldots,$ $D_y^T v_\ell (\ell \in L_0(\overline{x}^j, \overline{y}^j)),\, \ldots)^T$ are evaluated at (x,y), and $C = (\ldots,\, \xi_\eta^T\ (\eta \in \{1,\ldots,q-r-p^j\}),\ldots)$.

(b) Moreover, one can perform translations which transform the z-space $I\!R^q$ such that the finitely or countably many sets Z^j $(j \in J)$ become pairwise disjoint, with maybe noncompact union Z. Then, one can glue together the transformed functions g_j^0 $(j \in J)$ to real-valued C^1-functions $g_0^0 : I\!R^n \times I\!R^q \to I\!R$. In this way one could in $\overline{\mathcal{U}^0}$ equivalently express $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$, $\mathcal{P}^o_{\mathcal{SI}}(f,h,g^0,u^0,v^0)$ as an ordinary semi-infinite problem $\mathcal{P}^o_{\mathcal{SI}}(f,h,g_0^0,u^0,v^0)$ which has only one inequality constraint function on the "upper stage", but a maybe noncompact index set Z of inequality constraints.

2. Optimality conditions, a second equivalent model

With the preparations of Section 1, we are able to generalize the results on necessary or sufficient optimality conditions from Kaiser and Krabs ([28]). In fact, due to the case where $Y(x)$ is an interval $[a(x), b(x)] \subset I\!R$ $(q = 1)$, in [28] the optimality conditions for a generalized semi-infinite problem could be traced back to optimality conditions for an ordinary semi-infinite problem. Now, we can extend this tracing back for cases of higher dimensional manifolds $Y(x)$ with generalized boundary (cf. [16]). For ordinary semi-infinite optimization problems, optimality conditions have been worked out; cf. [13, 26]. While hereby in [26] a compactness assumption is made corresponding to our Assumption A, we may even use [26] for a noncompact fixed index set Y of inequality constraints. For this generalization we can replace the topology of uniform convergence on Y by the topology C^0_W of uniform convergence on all the compact subsets of Y. For more information on Whitney's weak topologies C^k_W $(k \in I\!N_0 := I\!N \cup \{0\})$ we refer to [14, 17].

We need some more notation. Whenever we disregard the inequality constraints $g(x,y) \geq 0$ $(y \in Y(x))$ then we denote the feasible set by $M[h]$. In [28], instead of $M[h]$ arbitrary nonempty sets $X \subseteq I\!R^n$ with convex tangent cone $T_{\hat{x}}X$ ([26]) at a minimum $\hat{x} \in X$ are considered. However, a theorem of Whitney (cf. [3], Theorem 3.3) tells us that each closed set $X \subseteq I\!R^n$ can be represented by means of a C^∞-function $\hat{h}$ as $X = M[\hat{h}]$.

Now, for each $\overline{x} \in M[h]$ at which LICQ holds,

$$T_{\overline{x}}\, M[h] \;:=\; \{\, \xi \in I\!R^n \mid\; Dh_i(\overline{x})\, \xi \;=\; 0 \;\; (i \in I)\,\} \tag{2.1}$$

stands for the tangent space at $\overline{x}$ on $M[h]$. If $\overline{x} \in M_{\mathcal{SI}}[h,g]$ this space contains the (linearized) tangent cone at $\overline{x}$ on $M_{\mathcal{SI}}[h,g]$:

$$C_{\overline{x}}\, M_{\mathcal{SI}}[h,g] \;:=\; \{\, \xi \in T_{\overline{x}}\, M[h] \;\mid\; D_x g_j^0(\overline{x},z)\, \xi \;\geq\; 0 \;\; ((j,z) \in Z_0^o(\overline{x}))\,\}. \tag{2.2}$$

Let $\overline{x} := \hat{x} \in M_{\mathcal{SI}}[h,g]$ be a local minimum for $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$ where LICQ is fulfilled at $\hat{x}$ as an element of $M[h]$ (in short: fulfilled at $\hat{x} \in M[h]$). Hence, we refer to all $x \in M_{\mathcal{SI}}[h,g] \cap \mathcal{U}^0$ being in competition, where $\mathcal{U}^0$ is a suitable neighborhood of $\hat{x}$. In the case $Z_0^o(\hat{x}) = \emptyset$ then we learn from [26], Section 3.1, that it holds

$$Df(\hat{x})\, \xi \;\geq\; 0 \quad \text{for all } \xi \in T_{\hat{x}}\, M[h]. \tag{2.3}$$

In the general case where $Z_0^o(\hat{x}) \neq \emptyset$ is admitted, but where moreover the (relatively) open tangent cone

$$C^*_{\hat{x}}\, M_{\mathcal{SI}}[h,g] \;:=\; \{\, \xi \in T_{\hat{x}}\, M[h] \;\mid\; D_x g_j^0(\hat{x},z)\, \xi \;>\; 0 \;\; ((j,z) \in Z_0^o(\hat{x}))\,\} \tag{2.4a}$$

is also $\neq \emptyset$, then we conclude with [26], Theorem III.3.5 and Lemma III.3.15:

$$\overline{C^*_{\hat{x}}\, M_{\mathcal{SI}}[h,g]} \;=\; C_{\hat{x}}\, M_{\mathcal{SI}}[h,g], \tag{2.4b}$$

$$Df(\hat{x})\, \xi \;\geq\; 0 \quad \text{for all } \xi \in C_{\hat{x}}\, M_{\mathcal{SI}}[h,g]. \tag{2.5}$$

The notations in (2.2), (2.4a) are justified by the local representation (1.9a).

Before we evaluate the necessary optimality condition (2.5) in the following result, let us recall the by x parametrized new local coordinates around $\overline{y}^j$ $(j \in J)$. They are of the form $\phi_x^j(y) = \widehat{F}(x,y)$, with $\widehat{F} := \widehat{F}^j$ given by (1.3a) for $(\overline{x},\overline{y}) := (\overline{x}^j,\overline{y}^j)$ $(j \in J)$. By means of calculus an explicit representation of the D_x-derivative of the implicit function $\hat{y}^j(x,z) = (\phi_x^j)^{-1}(z)$ (cf. (1.5b)) can be found:

$$G^j(x,z) \;:=\; D_x \hat{y}^j(x,z) =$$

$$= -\big(D_y \widehat{F}^j(x,(\phi_x^j)^{-1}(z))\big)^{-1} D_x \widehat{F}^j(x,(\phi_x^j)^{-1}(z)) \quad ((x,z) \in \mathcal{U}_1^{\overline{a}^j} \times \mathcal{C}_2^j). \tag{2.6}$$

The derivatives on the right hand side are well known (cf. (1.3a), (1.4b)). In particular, the last $q - r - p^j$ components of $D_x \widehat{F}^j$ vanish. With these explanations for a further evaluation and with the definition of the problem $\mathcal{P}^o_{\mathcal{SI}}(f,h,g,u^0,v^0)$, now we state:

Theorem 2.1 (Theorem on a necessary optimality condition (N1)). *Let $\hat{x} \in M_{\mathcal{SI}}[h,g]$ be a local minimum for the generalized semi-infinite optimization problem $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$, say: minimal on $M_{\mathcal{SI}}[h,g] \cap \mathcal{U}^0$ where $\mathcal{U}^0$ is some open neighborhood of $\hat{x}$, and let Assumption $B_{\mathcal{U}^0}$ hold. Moreover, let LICQ be fulfilled at $\hat{x} \in M[h]$ and the (relatively open linearized) tangent cone $C^*_{\hat{x}}\, M_{\mathcal{SI}}[h,g]$ be nonempty.*

Then, referring to canonical C^1-smooth local changes ϕ_x^j of the coordinates of y, we have

$$Df(\hat{x})\, \xi \geq 0 \tag{2.7}$$

for all $\xi \in I\!R^n$ *with*

$$Dh_i(\hat{x})\,\xi = 0 \quad for\ all \quad i \in I, \tag{2.8}$$

$$\big(D_x g(\hat{x}, (\phi^j_{\hat{x}})^{-1}(z)) + D_y g(\hat{x}, (\phi^j_{\hat{x}})^{-1}(z)) G^j(\hat{x}, z)\big)\,\xi \ \geq\ 0 \quad for\ all \quad (j,z) \in Z^o_0(\hat{x}). \tag{2.9}$$

We note that, on the one hand, (2.9) can also be expressed in the original variable y, namely referring to active indices $y = (\phi^j_x)^{-1}(z) \in Y_0(\hat{x})$.

On the other hand, (2.9) precisely means $D_x g^0_j(\hat{x}, z)\,\xi \geq 0$ for all $(j,z) \in Z^o_0(\hat{x})$. Then, we learn from [26], Theorem III.3.16 and what follows, in the presence of equality constraints, under the assumptions of Theorem 2.1 and Assumption $\mathrm{A}_{\mathcal{U}^0}$, that the implication of the previous theorem can equivalently be expressed as the following Kuhn — Tucker condition. Namely, with $I\!R_+$ denoting the set of nonnegative real numbers and referring to the gradients we have

$$(\mathrm{KT})^{\mathrm{o}} \quad \begin{cases} \text{There is a finite subset } Z^{fi}_0(\hat{x}) := \{\, (j^\kappa, z^\kappa) \mid \kappa \in \{1,\ldots,\hat{\kappa}\}\,\} \text{ of } Z^o_0(\hat{x}) \\ \text{and numbers } \lambda_i \in I\!R\ (i \in I),\ \mu_\kappa \in I\!R_+\ (\kappa \in \{1,\ldots,\hat{\kappa}\}), \text{ where } \hat{\kappa} \in I\!N_0, \\ \text{such that} \\ \\ Df(\hat{x}) \ =\ \sum_{i=1}^m \lambda_i\, Dh_i(\hat{x}) \ +\ \sum_{\kappa=1}^{\hat{\kappa}} \mu_\kappa\, D_x g^0_{j^\kappa}(\hat{x}, z^\kappa). \end{cases}$$

For our local minimum $\hat{x}$ this conclusion can also be attained by means of [13], Satz 3.1.14b) (see also [30]). Indeed, Assumption $\mathrm{B}_{\mathcal{U}^0}$, LICQ for $\hat{x} \in M[h]$ together with $C^*_{\hat{x}}\, M_{\mathcal{SI}}[h,g] \neq \emptyset$ precisely means the extended Mangasarian — Fromovitz constraint qualification EMFCQ ([13, 21, 32]) at $\hat{x}$ as an element of $M^o_{\mathcal{SI}}[h, g^0]$. The finite version of EMFCQ, called MFCQ, will be introduced below.

In view of (1.3a), (2.6), (2.9), of the chain rule and of the choice of $\overline{a}^j$ $(j \in J)$, $(\mathrm{KT})^o$ can further be evaluated. In this way we get a Kuhn — Tucker theorem on our local minimum $\hat{x}$ of $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$, which can also be proved by means of the Kuhn — Tucker theorem of Jongen, Rückmann and Stein ([19]). Actually, provided that

$$\text{for each } (j,z) \in Z^o_0(\hat{x}) \text{ the point } z \text{ does not belong to a (boundary) face} \\ \{\, z \in Z^j \mid z_\kappa = b^j_{\kappa - r} \,\},\ \{\, z \in Z^j \mid z_\sigma = 0 \,\}\ \ (\kappa \in \{r+1,\ldots,q\},\ \sigma \in \{r+p^j+1,\ldots,q\}) \text{ of } Z^j,$$

and, implicitly referring to the set $Z^{fi}_0(\hat{x})$ from $(\mathrm{KT})^o$, then we have:

$$(\overline{\mathrm{KT}}) \quad \begin{cases} \text{There is a finite subset } Y^{fi}_0(\hat{x}) := \{\, y^\kappa \mid\ y^\kappa = (\phi^{j^\kappa}_{\hat{x}})^{-1}(z^\kappa),\ \kappa \in \{1,\ldots,\hat{\kappa}\}\,\} \\ \text{of } Y_0(\hat{x}) \text{ and numbers } \lambda_i \in I\!R,\ \ \mu_\kappa \in I\!R_+,\ \alpha_{\kappa,k} \in I\!R,\ \beta_{\kappa,\ell} \in I\!R_+ \\ (i \in I,\ k \in K,\ \ell \in L_0(\hat{x}, y^\kappa),\ \kappa \in \{1,\ldots,\hat{\kappa}\}), \text{ where } \hat{\kappa} \in I\!N_0, \\ \text{such that} \\ \\ Df(\hat{x}) \ =\ \sum_{i=1}^m \lambda_i\, Dh_i(\hat{x}) \ +\ \sum_{\kappa=1}^{\hat{\kappa}} \mu_\kappa\, D_x g(\hat{x}, y^\kappa)\ - \\ \qquad -\ \sum\limits_{\substack{k \in \{1,\ldots,r\} \\ \kappa \in \{1,\ldots,\hat{\kappa}\}}} \alpha_{\kappa,k}\, D^T_x u_k(\hat{x}, y^\kappa) \ -\ \sum\limits_{\substack{\ell \in L_0(\hat{x}, y^\kappa) \\ \kappa \in \{1,\ldots,\hat{\kappa}\}}} \beta_{\kappa,\ell}\, D^T_x v_\ell(\hat{x}, y^\kappa). \end{cases}$$

Indeed, in virtue of our boundary condition, each active inequality constraint on z (in new variables) always represents an active (original) inequality constraint on y. Within the context of both our local linearizations (LICQ) and under the boundary condition from above, we also refer to [15] for Kuhn — Tucker conditions from finitely constrained optimization. In particular, there we learn the nonnegativity of the "Lagrange multipliers" $\beta_{\kappa,\ell}$, being $\dfrac{\partial}{\partial z_{r+\omega}} g^0_{j^\kappa}(\hat{x}, z^\kappa)$ up to the factor μ_κ ($\omega = \omega^\kappa$, $\ell = \ell^{\omega^\kappa} \in L_0(\hat{x}, y^\kappa)$, $\kappa \in \{1, \ldots, \hat{\kappa}\}$; cf. (1.3a)). Moreover, that boundary condition on the geometry of $Z^o_0(\hat{x})$ is the content of Assumption F which will lateron (in the case $K = \emptyset$) together with that nonnegativity be introduced. Let us already note that, by definition of Z^j ($j \in J$), our condition (that assumption) can also easily (but nonlinearly) be expressed in the original coordinates of y.

Example 2.2 (cf. [28]). Let us turn to the problem formulation from [28] being motivated from a mechanical engineering model (see Section 1), where we still assume all defining data to be of class C^1. Then we are in the special case where $K = \emptyset$, $Y(x) = [a(x), b(x)]$, say, in our formulation with (maybe) $I \neq \emptyset$, for all $x \in I\!R^n$ and where $a(x) < b(x)$ for all $x \in \overline{\mathcal{U}}$, $\mathcal{U} \subset I\!R^n$ being a (possibly bounded) neighborhood of $M_{\mathcal{SI}}[h, g]$. Consequently, we can easily choose a new coordinate z by means of parametrizing the interval $[a(x), b(x)]$. Hence, we get $z = \phi_x(y)$, with $\phi_x^{-1}(z) = a(x) + z \cdot (b(x) - a(x))$ $(w = 1)$ and the new index set $Z = [0, 1]$. In this easy example, the special form of $Y(x)$ guarantees LICQ and, hence, Assumption B to be fulfilled. However, the diffeomorphic representation of $Y(x) = M_{\mathcal{F}}[v(x, \cdot)]$, with $v_1(x, y) = -a(x) + y$, $v_2(x, \cdot) = b(x) - y$, performed in the general way of Section 1, would lead to a bit more notation. Hereby, we would refer to mappings $(\phi^1_x)^{-1}(z) = z + a(x)$, $(\phi^2_x)^{-1}(z) = -z + b(x)$.

Now, with our preferred new coordinate, (2.9) can be subdivided by our case study as follows:

$$\left(D_x g(\hat{x}, a(\hat{x}) + z \cdot (b(\hat{x}) - a(\hat{x}))\right) \xi \;\geq\; 0 \qquad \text{for all} \quad z \in Z^o_0(\hat{x}) \cap (0, 1) \tag{2.10a}$$

where the additional shift term from (2.9) vanishes, and

$$\begin{aligned} \big(D_x g(\hat{x}, a(\hat{x})) + D_y g(\hat{x}, a(\hat{x})) Da(\hat{x})\big)\, \xi \geq 0 \qquad &\text{if} \quad 0 \in Z^o_0(\hat{x}), &(2.10\text{b})\\ \big(D_x g(\hat{x}, b(\hat{x})) + D_y g(\hat{x}, b(\hat{x})) Db(\hat{x})\big)\, \xi \geq 0 \qquad &\text{if} \quad 1 \in Z^o_0(\hat{x}). &(2.10\text{c}) \end{aligned}$$

In order to realize the sufficiency for (local or even global) optimality of the condition (2.7) for all $\xi \in I\!R^n$ with (2.8), (2.9), we may suitably take over from [28] three more assumptions into our model. Hereby, we refer to a given point $\hat{x} \in M_{\mathcal{SI}}[h, g]$. For the conditions involved in the Assumptions D and E we refer to [26].

Assumption C. *The set* $M[h]$ is star-shaped *with* $\hat{x}$ *being a star point, i. e.*

$$\hat{x} + \lambda \cdot (x - \hat{x}) \in M[h] \qquad \textit{for all } x \in M[h],\ \lambda \in [0, 1].$$

Assumption D. *For all* $j \in J$, $z \in Z^j$, *the functions* $g^0_j(\cdot, z)$ *are quasi-concave on* $M[h]$ *with respect to* $\hat{x}$; *i. e., for each* $x \in M[h]$ *the following implication holds:*

$$g^0_j(x, z) \;\geq\; g^0_j(\hat{x}, z) \quad \Longrightarrow \quad D_x g^0_j(\hat{x}, z)\,(x - \hat{x}) \;\geq\; 0. \tag{2.11}$$

The implication (2.11) can be rewritten in the original data by writing the right hand side as the inequality from (2.9), whereby $\xi = x - \hat{x}$.

Assumption E. *The function* f *is pseudo-convex on* $M[h]$ *with respect to* $\hat{x}$;
i. e., for each $x \in M[h]$ *the following implication holds:*

$$Df(\hat{x})\,(x - \hat{x}) \;\geq\; 0 \quad \Longrightarrow \quad f(x) \geq f(\hat{x}). \tag{2.12}$$

We also introduce the corresponding local versions Assumptions $\mathrm{C}_{\mathcal{U}^0}$, $\mathrm{D}_{\mathcal{U}^0}$, $\mathrm{E}_{\mathcal{U}^0}$. Therefore, $M[h]$ becomes replaced by $M[h] \cap \overline{\mathcal{U}^0}$, where $\mathcal{U}^0 \subseteq I\!R^n$ is open.

For more information on quasi-concavity, pseudo-convexity and related conditions in the contexts of optimization (inf-compactness and solutions) and of differential geometry (implicit or parametrized surfaces), we refer to [5] and [6], respectively. The development of iteration procedures underlines the practical importance of our different assumptions ([32]).

Example 2.3 (cf. [28]). Assumption C is fulfilled if, e. g., $M[h]$ is convex, and it implies arc-wise connectedness of $M[h]$. Without severe restrictions we may think $M[h] \subset \mathcal{U}$.

The following example (due to [28]) for the Assumptions D, E continues Example 2.2 for the convex set $M[h]$: $g(x,y) := g_1(y) + g_2(x)$ with g_1, g_2 being concave, f is convex, and a, b are affinely linear. These properties may hold globally, or, referring to a further (possibly bounded) open set $\mathcal{V} \subset I\!R$, for all $x \in \mathcal{U}$, $y \in \mathcal{V}$, where $Y(x) \subset \mathcal{V}$ $(x \in \overline{\mathcal{U}})$. Note, that then $g^0(x,z) = g(x, \phi_x^{-1}(z)) = g_1(a(x) + z \cdot (b(x) - a(x))) + g_2(x)$ is concave in x such that $g^0(x,z) - g^0(\tilde{x},z) \leq D_x g^0(\tilde{x},z)\,(x-\tilde{x})$, and $f(x) - f(\tilde{x}) \geq Df(\tilde{x})\,(x-\tilde{x})$ for all x, $\tilde{x}$. We choose $\tilde{x} := \hat{x}$.

In this example, the local versions of our assumptions can easily be given, too.

Theorem 2.4 on a sufficient optimality condition (S1)[3]. *(a) Let the Assumption B hold for $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$. Moreover, for some given $\hat{x} \in M_{\mathcal{SI}}[h,g]$ we assume LICQ for $\hat{x} \in M[h]$ and the Assumptions C, D, E to be fulfilled.*

If, moreover, the condition (2.7) holds for each $\xi \in I\!R^n$ with (2.8), (2.9), then $\hat{x}$ is a global minimum for $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$.

(b) In (a) we replace the Assumptions B – E by the Assumptions $B_{\mathcal{U}^0}$ – $E_{\mathcal{U}^0}$ where $\mathcal{U}^0$ is some open neighborhood of the point $\hat{x}$. Then, under the further assumption of LICQ at $\hat{x} \in M[h]$ and of (2.7) for all $\xi \in I\!R^n$ with (2.8), (2.9), $\hat{x}$ turns out to be a local minimum for $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$.

Proof. It is enough to demonstrate the first part of the theorem because then the second part immediately follows. Indeed, a global minimum for f on $M_{\mathcal{SI}}[h,g] \cap \mathcal{U}^0$ is a local minimum for $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$. Now, for each given $x \in M_{\mathcal{SI}}[h,g]$ we have to show $f(\hat{x}) \leq f(x)$.

From our Assumption C we conclude $x - \hat{x} \in T_{\hat{x}}\, M[h]$. This means the validity of (2.8) for $\xi := x - \hat{x}$. For each given $(j,z) \in Z_0^o(\hat{x})$ we have $g_j^0(x,z) - g_j^0(\hat{x},z) = g_j^0(x,z) \geq 0$. Then, for ξ the inequality from (2.9) holds because of Assumption D.

Now, as in the case of (2.8), (2.9), the inequality (2.7) holds by assumption, it follows $Df(\hat{x})\,(x-\hat{x}) \geq 0$. Finally, Assumption E allows us to state $f(\hat{x}) \leq f(x)$.

We introduce x-dependent subsets $M(x)$ $(x \in I\!R^n)$ of the feasible set $M_{\mathcal{SI}}[h,g]$ by restrictively implying the feasible sets $Y(x)$ of the "*lower stage*" (for a general introduction see [28]):

$$\left.\begin{array}{rcl} M(\tilde{x}) & = & \{\, x \in I\!R^n \mid \ Y(x) \subseteq Y(\tilde{x}), \\ & & \quad h_i(x) = 0 \ (i \in I), \ \ g(x,y) \geq 0 \ (y \in Y(\tilde{x}))\,\}. \end{array}\right\} \tag{2.13}$$

Lemma 2.5 on a necessary optimality condition (N2)[4]. *Let a local or global minimum $\hat{x}$ of $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$ be given. Then, it holds $\hat{x} \in M(\hat{x})$, and $\hat{x}$ is a local or global minimum for f on $M(\hat{x})$ respectively.*

Proof. We have the following representation of $M_{\mathcal{SI}}[h,g]$:

$$M_{\mathcal{SI}}[h,g] \ = \ \cup_{\tilde{x} \in M[h]}\, M(\tilde{x}). \tag{2.14}$$

[3] Cf. also [28], Satz 2.

[4] Cf. [28]

Namely, from (2.13) we conclude the implications

$$\begin{aligned} \tilde{x} \in M_{\mathcal{SI}}[h,g] \quad &\Longleftrightarrow \quad \tilde{x} \in M(\tilde{x}) \qquad (\tilde{x} \in M[h]), &(2.15a)\\ x \in M(\tilde{x}) \text{ for some } \tilde{x} \in M[h] \quad &\Longrightarrow \quad x \in M_{\mathcal{SI}}[h,g], &(2.15b)\end{aligned}$$

from which the inclusions "$\subseteq$, $\supseteq$" for (2.14) follow, respectively. In view of (2.15a), (2.14), a global minimum $\tilde{x} := \hat{x}$ belongs to $M(\hat{x})$ and it minimizes f on $M(\hat{x})$. For a local minimum $\hat{x}$ the corresponding assertions follow analogously, referring to some suitable neighborhood $\mathcal{U}^0$ of $\hat{x}$.

In view of (2.14) and as far as the two conditions of being a local or global minimum are concerned, respectively, the reverse implication of Lemma 2.5 cannot hold in general.

For a given feasible (e. g., locally minimal) point $\hat{x}$ we can, after some preparations and assumptions, express the set $M(\hat{x})$ as a feasible set:

$$M(\hat{x}) \;=\; M_{\mathcal{SI},\hat{x}}[h, g^{\vee}], \tag{2.16a}$$

namely in the way of defining which is subsequently described and proved[5].

Now, firstly we replace each of our possible compactness assumptions by the following condition on global arc-wise connectedness (which can be interpreted as some "stiffness"):

Assumption A$^{\vee}$ (Connectedness). *There is a neighborhood $\mathcal{U}$ of $M_{\mathcal{SI}}[h,g]$ such that for all x^1, $x^2 \in \overline{\mathcal{U}}$ it holds $(Y(x^1))^{\circ} \cap (Y(x^2))^{\circ} \neq \emptyset$, and each of the sets $Y(x)$ $(x \in \overline{\mathcal{U}})$ is arc-wise connected.*

Hereby, $(Y(x))^{\circ}$ denotes the interior of $Y(x)$, relatively in $M[u(x,\cdot)]$ $(x \in \overline{\mathcal{U}})$. Then, under our further Assumption B, referring to the open set $\mathcal{U}$ from Assumption A$^{\vee}$, we have the representation (cf. [15])

$$(Y(x))^{\circ} \;=\; \{\, y \in M[u(x,\cdot)] \mid \; \min_{\ell \in L} v_{\ell}(x,y) > 0 \,\} \;\subseteq\; Y(x) \qquad (x \in \overline{\mathcal{U}}). \tag{2.17a}$$

As a *local* version we introduce, referring to those relative topologies again,

Assumption A$^{\vee}_{\mathcal{U}^0}$ (Connectedness, locally). *Referring to some open set $\mathcal{U}^0 \subseteq I\!R^n$, for each x^1, $x^2 \in \overline{\mathcal{U}^0}$ the arc components $\mathcal{K}^1_{\gamma^1}$, $\mathcal{K}^2_{\gamma^2}$ $(\gamma^j \in \Gamma^j,\ j \in \{1,2\})$ of $Y(x^1), Y(x^2)$ pairwise correspond to each other in such a way that $\Gamma := \Gamma^1 = \Gamma^2$ and, pairwise, $\gamma := \gamma^1 = \gamma^2$ $(\gamma^j \in \Gamma,\ j \in \{1,2\})$, and, moreover, $(\mathcal{K}^1_{\gamma})^{\circ} \cap (\mathcal{K}^2_{\gamma})^{\circ} \neq \emptyset$ for all corresponding arc components $\mathcal{K}^1_{\gamma}$, $\mathcal{K}^2_{\gamma}$ of $Y(x^1)$ and $Y(x^2)$, respectively $(\gamma \in \Gamma)$.*

As for the purpose of our problem representation we may weaken the Assumptions B and B$_{\mathcal{U}^0}$, we introduce the following constraint qualification which is implied by LICQ (cf. [17, 29]):

Definition 2.6. Let points $\overline{x} \in I\!R^n$ and $\overline{y} \in Y(\overline{x})$ be given. We say that the *Mangasarian – Fromovitz constraint qualification*, in short: *MFCQ*, holds at $\overline{y}$ as an element of the feasible set $M_{\mathcal{F}}[u(\overline{x},\cdot), v(\overline{x},\cdot)]$ if the following conditions are fulfilled:

MF1. The vectors $D_y u_k(\overline{x},\overline{y})$ $(k \in K)$ are linearly independent.

MF2. There exists a vector $\xi \in I\!R^n$ satisfying

$$D_y u_k(\overline{x},\overline{y})\,\xi = 0 \quad (k \in K),$$

$$D_y v_{\ell}(\overline{x},\overline{y})\,\xi > 0 \quad (\ell \in L_0(\overline{x},\overline{y})).$$

Such a vector ξ is called an *MF-vector*.

[5]If the reader is not so interested in the technical details, then he might after a short study of the assumptions which follow, of (2.22) and Definition 2.6, more directly turn to Theorem 2.7.

The *Mangasarian — Fromovitz constraint qualification (MFCQ)* is said to hold for the feasible set $M_{\mathcal{F}}[u(\overline{x},\cdot),v(\overline{x},\cdot)]$ if MFCQ is fulfilled for all its elements $y \in Y(\overline{x})$.

Assumption $\widetilde{\mathrm{B}}$, $\widetilde{\mathrm{B}}_{\mathcal{U}^0}$ (MFCQ, globally or locally). *In the Assumptions B, $B_{\mathcal{U}^0}$ we replace LICQ by MFCQ.*

Whenever we have $K = \emptyset$, Assumption $\mathrm{A}_{\mathcal{U}^0}$ being fulfilled and $\mathcal{U}^0$ being a sufficiently small neighborhood of $\hat{x}$, then Assumption $\mathrm{B}_{\mathcal{U}^0}$, or $\widetilde{\mathrm{B}}_{\mathcal{U}^0}$, already implies Assumption $\mathrm{A}^{\vee}_{\mathcal{U}^0}$. From [8,16] we can learn that under the Assumptions $\mathrm{A}_{\mathcal{U}^0}$, $\mathrm{B}_{\mathcal{U}^0}$, moreover, Γ is of finite cardinality.

Therefore, maybe we have to turn to a smaller neighborhood $\mathcal{U}^1 \subset \mathcal{U}^0$. If however, $\mathcal{U}^0$ (or, globally, $\mathcal{U}$), considered as a bounded parameter set, is arc-wise connected, too, then such a shrinkening is not necessary.

Indeed, then our correspondences can be expressed by means of global homeomorphisms. For this implication which in fact does not need LICQ but MFCQ, we refer to [9, 21]. Moreover, for some related situation on the upper level, our implication of homeomorphical correspondence is stated in [32].

Finally, that special parametrical aspect on the presence of arc-wise connectedness is (for one parameter) given in [22].

In order firstly to give a global problem discussion and for the ease of exposition, we begin with making the Assumptions $\mathrm{A}^{\vee}$, $\widetilde{\mathrm{B}}$. Lateron, however, we shall see and discuss that locally the Assumptions $\mathrm{A}^{\vee}_{\mathcal{U}^0}$ and $\mathrm{B}_{\mathcal{U}^0}$ (or $\widetilde{\mathrm{B}}_{\mathcal{U}^0}$) are appropriate for the goals of representation, optimality conditions and, in [32], convergence of iteration procedures.

Now, we introduce the following defining inequality constraint functions:

$$\left.\begin{array}{ll} g^{\vee} = (g^{\vee}_1, g^{\vee}_2, g^{\vee}_3), & \text{where} \\ \quad g^{\vee}_1 := g, & \\ g^{\vee}_2 = (g^{\vee}_{2,1},\ldots,g^{\vee}_{2,r}), & g^{\vee}_{2,k}(x,y) := -u^2_k(\hat{x},y) \quad (k \in K), \\ g^{\vee}_3 = (g^{\vee}_{3,1},\ldots,g^{\vee}_{3,s}), & g^{\vee}_{3,\ell}(x,y) := -v_\ell(x,y) \quad (\ell \in L) \end{array}\right\} \tag{2.18}$$

and the corresponding index sets

$$Y^{\vee 1} := Y(\hat{x}), \quad Y^{\vee 2,k}(x) := Y(x) \ (x \in I\!R^n,\ k \in K), \quad Y^{\vee 3,\ell} := Y(\hat{x}) \cap Y^{\ell}_0(\hat{x}) \ (\ell \in L). \tag{2.19a}$$

Hereby, for each $\ell \in L$ the definition

$$Y^{\ell}_0(\hat{x}) \ := \ \{ y \in I\!R^q \mid v_\ell(\hat{x},y) = 0 \} \tag{2.19b}$$

means that $Y^{\vee 3,\ell}$ comes from $Y(\hat{x}) = M_{\mathcal{F}}[u(\hat{x},\cdot),v(\hat{x},\cdot)]$ by deleting $v_\ell(\hat{x},\cdot)$ as an inequality constraint, but by treating $v_\ell(\hat{x},\cdot)$ as an equality constraint. In this sense we may with the help of suitable defining functions also write

$$Y^{\vee 3,\ell} \ := \ M_{\mathcal{F},\hat{x}}[u^{\vee \ell}, v^{\vee \ell}] \quad (\ell \in L). \tag{2.19c}$$

In the special case where Assumption B is fulfilled we may state that all the feasible sets given in (2.19a) fulfill LICQ. Hence, they are manifolds with generalized boundaries (cf. [16]). However, if only Assumption $\widetilde{\mathrm{B}}$ holds, then these sets need not all to fulfill MFCQ (namely, consider $Y^{\vee 3,\ell}$ $(\ell \in L)$). But if they fulfill MFCQ, then they are manifolds with Lipschitzian boundaries (cf. [9, 21, 30]). The same statements can also locally be made referring to $\mathrm{B}_{\mathcal{U}^0}$ and $\widetilde{\mathrm{B}}_{\mathcal{U}^0}$, respectively.

Moreover, let us for each $x \in \overline{\mathcal{U}}$ denote the corresponding "active" subsets by

$$Y_0^{\vee 1}(x), \quad Y_0^{\vee 2,k}(x)\ (k \in K), \quad Y_0^{\vee 3,\ell}(x)\ \ (\ell \in L), \tag{2.20a}$$

where

$$\begin{aligned} Y_0^{\vee 2,k}(x) &:= \{\, y \in Y(x) \,|\, u_k(\hat{x}, y) = 0 \,\} \quad (k \in K), &(2.20b)\\ Y_0^{\vee 3,\ell}(x) &:= \{\, y \in Y(\hat{x}) \,|\, v_\ell(\hat{x}, y) = v_\ell(x, y) = 0 \,\} \quad (\ell \in L), &(2.20c)\end{aligned}$$

and, without misunderstandings using the index j also in the second approach, namely for the following enumerated union:

$$Y_0^{\vee}(x) \;=\; \{\, (j, y) \;|\; y \in \left\{ \begin{array}{ll} Y_0^{\vee 1}(x), & \text{if } j = 1 \\ Y_0^{\vee 2,k}(x), & \text{if } j = 2 \\ Y_0^{\vee 3,\ell}(x), & \text{if } j = 3 \end{array} \right\}, \ \text{ for some } k \in K,\ \ell \in L \,\}. \tag{2.21}$$

Let the functions defining the index sets of (2.19a) systematically be ordered in a way being compatible with the indices from (2.21). Then, we comprise them by $u^{\vee}$ and $v^{\vee}$, respectively; cf. (2.19c) for some finite subfamilies of defining component functions.

Now, we can give the following proof of (2.16a) and, hence, we may represent the minimization problem for f on $M(\hat{x})$ by

$$\mathcal{P}_{\mathcal{SI},\hat{x}}(f, h, g^{\vee}, u^{\vee}, v^{\vee}) \quad \left\{ \begin{array}{r} \text{Minimize } f(x) \text{ on } M_{\mathcal{SI},\hat{x}}[h, g^{\vee}], \quad \text{where} \\[2ex] M_{\mathcal{SI},\hat{x}}[h, g^{\vee}] \;:=\; \{\, x \in I\!R^n \;|\; h_i(x) = 0 \ \ (i \in I), \\[2ex] g_1^{\vee}(x, y) \geq 0 \ \ (y \in Y^{\vee 1}), \\[2ex] g_{2,k}^{\vee}(x, y) \geq 0 \ \ (y \in Y^{\vee 2,k}(x),\ k \in K), \\[2ex] g_{3,k}^{\vee}(x, y) \geq 0 \ \ (y \in Y^{\vee 3,k},\ \ell \in L) \,\}. \end{array} \right\} \tag{2.22}$$

Proof of (2.16a). The last $r{+}s$ inequality constraints in (2.22) precisely reflect the inclusion $Y(x) \subseteq Y(\hat{x})$. Hereby, the implication "$\Longleftarrow$" of this equivalence is not hard to realize; let us turn to "$\Longrightarrow$". Therefore, we note in the following indirect way, that the Assumptions $\mathrm{A}^{\vee}$, $\widetilde{\mathrm{B}}$ do not allow some $y \in Y(x)$ fulfilling these inequalities, say $y \in (Y(x))^{\circ}$, to lie outside of $Y(\hat{x})$. This (relative) interior position can be guaranteed by means of a small inward shift of y. Hereby, we note that because of Assumption $\widetilde{\mathrm{B}}$ on MFCQ, $Y(x)$ is a manifold with Lipschitzian boundary (cf. [9]).

Otherwise, as in view of the definition of $g_{2,k}^{\vee}$ $(k \in K)$ the constraints $u_k = 0$ do not cause difficulties and as by Assumption $\mathrm{A}^{\vee}$ there is also a point $y^0 \in (Y(x))^{\circ} \cap (Y(\hat{x}))^{\circ}$, there exists an arc C in $Y(x)$ connecting y with y^0. Using the topological structure of $Y(x)$ again, we may say: $C \subseteq (Y(x))^{\circ}$. This arc has to meet the (relative) boundary $\partial Y(\hat{x})$ of $Y(\hat{x})$ in $M[u(\hat{x}, \cdot)]$ at a point y^*. Because of Assumption $\widetilde{\mathrm{B}}$ we have the representation (cf. [9], Theorem A)

$$\partial Y(\hat{x}) \;=\; \{\, y \in M[u(\hat{x}, \cdot)] \,|\; \min_{\ell \in L} v_\ell(\hat{x}, y) = 0 \,\} \;\subseteq\; Y(\hat{x}). \tag{2.17b}$$

Hence, there is an index $\ell^* \in L$ with $v_{\ell^*}(\hat{x}, y^*) = 0$, such that we conclude $g_{3,\ell^*}^{\vee}(x, y^*) \geq 0$, i. e.

$$v_{\ell^*}(x, y^*) \;\leq\; 0. \tag{2.23a}$$

From $y^* \in C \subseteq (Y(x))^\circ$, however, it follows

$$v_{\ell^*}(x, y^*) \ > \ 0, \tag{2.23b}$$

in contradiction with (2.23a).

Our problem $\mathcal{P}_{\mathcal{SI},\hat{x}}(f, h, g^\vee, u^\vee, v^\vee)$ is of generalized semi-infinite type such that under suitable assumptions (especially Assumption $\mathrm{B}_{\mathcal{U}^0}$) the results given in the Theorems 2.1 and 2.4 with their necessary and sufficient optimality conditions, namely N1. and S1., could easily be formulated. However, if there are no equality constraints on the lower stage of the original problem, i. e. if $K = \emptyset$, then we have turned to an ordinary semi-infinite optimization problem, called

$$\mathcal{P}^o_{\mathcal{SI},\hat{x}}(f, h, g^\vee, v^\vee)\,.$$

Furthermore, then, its feasible set and its active index sets are denoted by

$$M^o_{\mathcal{SI},\hat{x}}[h, g^\vee] := M_{\mathcal{SI},\hat{x}}[h, g^\vee], \qquad Y_0^o(x) := Y_0^\vee(x) \ \ (x \in M^o_{\mathcal{SI},\hat{x}}[h, g^\vee]).$$

Making the Assumptions $\mathrm{A}^\vee_{\mathcal{U}^0}$ (or $K = \emptyset$, $\mathrm{A}_{\mathcal{U}^0}$ and shrinkening $\mathcal{U}^0$, if necessary) and $\mathrm{B}_{\mathcal{U}^0}$ (or $\widetilde{\mathrm{B}}_{\mathcal{U}^0}$), our considerations remain valid. Hereby, our indirect argumentation refers to some pair $(\mathcal{K}^1_{\gamma^0}, \mathcal{K}^2_{\gamma^0})$ of corresponding components with $y \in \mathcal{K}^1_{\gamma^0}$, $y^0 \in \mathcal{K}^1_{\gamma^0} \cap \mathcal{K}^2_{\gamma^0}$. In particular, we have

$$M(\hat{x}) \cap \overline{\mathcal{U}^0} \ = \ M_{\mathcal{SI},\hat{x}}[h, g^\vee] \cap \overline{\mathcal{U}^0}, \qquad \text{and} \quad = M^o_{\mathcal{SI},\hat{x}}[h, g^\vee] \cap \overline{\mathcal{U}^0} \ \text{ if } K = \emptyset. \tag{2.16b}$$

Now, let us dispense with our goal of global representation, but turn to the *local* model in the case $K = \emptyset$.

With the help of the considerations at the beginning of this section and by means of Lemma 2.5, we may state now:

Theorem 2.7 on a necessary optimality condition (N3). *Let $\hat{x} \in M_{\mathcal{SI}}[h, g]$ be a local minimum for the generalized semi-infinite optimization problem $\mathcal{P}_{\mathcal{SI}}(f, h, g, v)$ ($K = \emptyset$), say: minimal on $M_{\mathcal{SI}}[h, g] \cap \mathcal{U}^0$ where $\mathcal{U}^0$ is some open neighborhood of $\hat{x}$, and let the Assumptions $A^\vee_{\mathcal{U}^0}$, or $A_{\mathcal{U}^0}$, and $\widetilde{B}_{\mathcal{U}^0}$ hold.*

Moreover, we assume LICQ to be fulfilled at $\hat{x} \in M[h]$ and the (relatively open linearized) tangent cone $C^*_{\hat{x}}\, M^o_{\mathcal{SI},\hat{x}}[h, g^\vee]$ to be nonempty.

Then, referring to the ordinary semi-infinite optimization problem $\mathcal{P}^o_{\mathcal{SI},\hat{x}}(f, h, g^\vee, v^\vee)$ we have

$$Df(\hat{x})\, \xi \ \geq \ 0 \tag{2.24}$$

for all $\xi \in \mathbb{R}^n$ with

$$D_x h_i(\hat{x})\, \xi \ = \ 0 \qquad \textit{for all } i \in I, \tag{2.25}$$

$$D_x g_j^\vee(\hat{x}, y)\, \xi \ \geq \ 0 \qquad \textit{for all } (j, y) \in Y_0^o(\hat{x}). \tag{2.26}$$

Hereby, (2.26) can equivalently be formulated as

$$D_x g(\hat{x}, y)\, \xi \ \geq \ 0 \qquad \textit{for all } y \in Y_0(\hat{x}), \tag{2.27a}$$

$$D_x v_\ell(\hat{x}, y)\, \xi \ \leq \ 0 \qquad \textit{for all } y \in Y(\hat{x}), \ \ell \in L_0(\hat{x}, y). \tag{2.27b}$$

Because of the absence of an intrinsic diffeomorphism, (2.26) does not reveal an (additional) shift-term as it is given in (2.9).

As we did for Theorem 2.1, under the assumptions of Theorem 2.7 and Assumption $\mathrm{A}_{\mathcal{U}^0}$ we can again express the implication of the previous theorem as a Kuhn—Tucker condition.

Namely, this time on the one hand we have a condition $(\mathrm{KT})^{\vee}$ being analogous with $(\mathrm{KT})^{o}$ for $K = \emptyset$. Hereby, $g^{\vee}$ takes over the part of g^0. On the other hand we realize the following specification of $(\overline{\mathrm{KT}})$ for the case $K = \emptyset$, where for simplicity we use some old notations again:

$$(\mathrm{KT}) \quad \left\{ \begin{array}{c} \text{There are finite subsets } Y_0^{fi}(\hat{x}) \;=\; \{\, y^{\kappa} \mid \kappa \in \{1, \dots, \hat{\kappa}\} \,\} \;\subseteq\; Y_0(\hat{x}), \\ Y^{fi}(\hat{x}) \;=\; \{\, y'^{\kappa'} \mid \kappa' \in \{1, \dots, \hat{\kappa}'\} \,\} \;\subseteq\; Y(\hat{x}), \ \text{ and numbers } \lambda_i \in I\!R, \\ \mu_{\kappa} \in I\!R_+, \ \beta_{\kappa',\ell} \in I\!R_+ \ \ (i \in I, \ \ell \in L_0(\hat{x}, y'^{\kappa'}), \ \kappa \in \{1, \dots, \hat{\kappa}\}, \ \kappa' \in \{1, \dots, \hat{\kappa}'\}) \\ \text{where } \hat{\kappa}, \, \hat{\kappa}' \in I\!N_0, \ \text{ such that} \\ Df(\hat{x}) \;=\; \sum_{i=1}^{m} \lambda_i \, Dh_i(\hat{x}) \;+\; \sum_{\kappa=1}^{\hat{\kappa}} \mu_{\kappa} \, D_x g(\hat{x}, y^{\kappa}) \;-\; \sum_{\substack{\ell \in L_0(\hat{x}, y'^{\kappa'}) \\ \kappa' \in \{1, \dots, \hat{\kappa}'\}}} \beta_{\kappa',\ell} \, D_x v_{\ell}(\hat{x}, y'^{\kappa'}). \end{array} \right.$$

Provided that the extended Mangasarian—Fromovitz constraint qualification of Jongen, Rückmann and Stein ([19]) holds, this Kuhn—Tucker result on a local minimum $\hat{x}$ of $\mathcal{P}_{\mathcal{SI}}(f, h, g, v)$ also follows from the Kuhn—Tucker theorem of [19]. Now, we may realize that for $\mathcal{P}^{o}_{\mathcal{SI},\hat{x}}(f, h, g^{\vee}, v^{\vee})$ neither the ambiguity nor the disadvantage exists, which were remarked for the problem $\mathcal{P}^{o}_{\mathcal{SI}}(f, h, g^0, u^0, v^0)$ of the first approach (Section 1).

However, now, of course $K = \emptyset$ means a restriction of the generality. Moreover, in Theorem 2.7 there are more inequality constraints involved into the nonemptiness condition on the relatively open linearized tangent cone than in Theorem 2.1.

Example 2.8 (cf. [28]). Continuing Example 2.2 (and, hence, Example 2.3) where we are in the case $K = \emptyset$, (2.27b) can be written as follows:

$$Da(\hat{x})\, \xi \;\geq\; 0 \qquad \text{and} \qquad Db(\hat{x})\, \xi \;\leq\; 0. \tag{2.28}$$

Here, we are in the special case $\underline{q = 1}$ which will become important below.

Let us for $\mathcal{P}_{\mathcal{SI}}(f, h, g, v)$ compare our necessary optimality conditions, namely N1. with N3., in particular, (2.9) with (2.27a,b). Therefore, we make the Assumptions $\mathrm{A}^{\vee}_{\mathcal{U}^0}$, or $\mathrm{A}_{\mathcal{U}^0}$, and $\mathrm{B}_{\mathcal{U}^0}$, where $\mathcal{U}^0$ is some open neighborhood of $\hat{x}$. Using (2.6), here we may express (2.9) as

$$D_x g(\hat{x}, (\phi^j_{\hat{x}})^{-1}(z))\, \xi \;+ D_z g^0_j(\hat{x}, z) \begin{pmatrix} \vdots \\ -D_x v_{\ell}(\hat{x}, (\phi^j_{\hat{x}})^{-1}(z)) \ \ (\ell \in L_0(\hat{x}, (\phi^j_{\hat{x}})^{-1}(z))) \\ \vdots \\ O_{(q-p^j) \times n} \end{pmatrix} \xi \;\geq\; 0$$

$$((j, z) \in Z^{o}_0(\hat{x})). \tag{2.29}$$

For each given index $(j, \overline{z}) \in Z^{o}_0(\hat{x})$ we know that $\overline{z}$ is a local minimum for $g^0_j(\hat{x}, \cdot)$ on $Z^j \subseteq I\!R^q$. Recall that Z^j is an axis-parallel cube of the form (see (1.8), where r may also be positive)

$$Z^j \;=\; [0, b^j_1] \times \cdots \times [0, b^j_q] \;\subseteq\; I\!H^{p^j} \times I\!R^{q-p^j}, \quad p^j = |L_0(\hat{x}, \overline{y}^j)|.$$

If $\overline{z}$ is not lying on the relative boundary

$$\begin{array}{c} \partial_+ Z^j \;:=\; \{\, z \in Z^j \mid \ z_{\sigma} = b^j_{\sigma} \ \text{ for some } \sigma \in \{1, \dots, q\}, \\ \text{or } \ z_{\sigma} = 0 \ \text{ for some } \sigma \in \{p^j + 1, \dots, q\} \,\} \end{array}$$

of Z^j in $I\!H^{p^j} \times I\!R^{q-p^j}$ then for the local minimum $\overline{z}$ it holds

$$\frac{\partial}{\partial z_\ell} g_j^0(\hat{x}, \overline{z}) \geq 0 \qquad (\ell \in \{1, \ldots, q\}), \tag{2.30}$$

where for $\ell \in \{p^j + 1, \ldots, q\}$ we know that 0 is attained $(= 0)$. Let us put the set

$$Z^o_{0+}(\hat{x}) := \{(j, z) \in Z^o_0(\hat{x}) \mid z \notin \partial_+ Z^j\}. \tag{2.31}$$

Let us remark that this notation of a set can indeed, with r leading components 0 of the vectors z, immediately be generalized for the case $r > 0$ $(K \neq \emptyset)$.

Now, we make an assumption on a special fineness of the open coverings from Section 1:

Assumption F (Technical fineness). *It holds* $Z^o_{0+}(\hat{x}) = Z^o_0(\hat{x})$.

In order not to go too much into the technicalities, we only note that this assumption rules out both any kind of activity of originally free coordinates, and any negativity in (2.30) for indices $\overline{z} \in \partial_+ Z^j$, $\ell \in L_0(\hat{x}, (\phi^j_{\hat{x}})^{-1}(\overline{z}))$. Such a nonnegativity, coming say, from the orientation within our local linearization, would mean that a Lagrange multiplier $\tilde{\beta}_{\kappa,\ell} = \frac{\partial}{\partial z_\omega} g^0_j(\hat{x}, \overline{z})$ (≥ 0) on the lower stage (i. e., $\beta_{\kappa,\ell} = \mu_\kappa \cdot \tilde{\beta}_{\kappa,\ell}$, $j = j^\kappa$, $\ell = \ell^\omega$, in the sense of (1.3a), $(\overline{\mathrm{KT}})$; $r = 0$) becomes negative.

Now, if the inequalities (2.27a,b) hold then, in view of (2.30), the sum on the left hand side of (2.29) turns out to be sum of nonnegative numbers. Hence, (2.9) is satisfied. Hereby, it is even enough in (2.27b) to refer to $Y_0(\hat{x})$ instead of $Y(\hat{x})$.

Let us think about the reverse direction of this implication. If we are in the special case $q = 1$, if, moreover, (2.9) holds and the following technical condition

$$(\mathcal{TC}) \left\{ \begin{array}{l} \textit{for each } (j, z) \in Z^o_0(\hat{x}) \textit{ and each active } \ell \in L, \textit{ i. e. (by LICQ)} \{\ell\} = L_0(\hat{x}, y) \\ \textit{where } y = (\phi^j_{\hat{x}})^{-1}(z), \textit{ there is a multiplier } \chi = \chi^z_{j,\ell} < 0 \textit{ solving the equation} \\ \\ D_x g(\hat{x}, y) = \chi D_x v_\ell(\hat{x}, y), \end{array} \right. \tag{2.32}$$

is fulfilled, then the reverse implication holds due to each given index $(j, z) \in Z^o_0(\hat{x})$. Indeed, whenever there is a corresponding active index ℓ, from (2.32) we conclude the inequality

$$\left(\chi^z_{j,\ell} - \frac{\partial}{\partial z} g^0_j(\hat{x}, z)\right) D_x v_\ell(\hat{x}, y)\, \xi \;\geq\; 0$$

from which the inequality $D_x v_\ell(\hat{x}, y)\ \xi \ \leq \ 0$ follows by means of (2.30) and of $\chi^z_{j,\ell} < 0$. With the help of the previous inequality, of $\chi^z_{j,\ell} < 0$ and (2.32), we realize the validity of $D_x g(\hat{x}, y)\ \xi \geq 0$. Hence, we have concluded (2.27a), and (2.27b) with $Y(\hat{x})$ being substituted by its subset $Y_0(\hat{x})$. Of course, the same implication can be stated whenever there is no active index ℓ.

We consider the last reflections in the context of our necessary optimality conditions.

Lemma 2.9 (cf. also [28], Satz 4). *Let a point* $\hat{x} \in M_{\mathcal{SI}}[h, g]$ *be given for the problem* $\mathcal{P}_{\mathcal{SI}}(f, h, g, v)$ *and the Assumptions* $B_{\mathcal{U}^0}$, F *hold where* $\mathcal{U}^0$ *is some open neighborhood of* $\hat{x}$.

Then, the following relations hold between the necessary optimality conditions (N1.,3.):

(a) (2.7) for all $\xi \in I\!R^n$ *with (2.8), (2.9)* $\Longrightarrow$ *(2.7) for all* $\xi \in I\!R^n$ *with (2.8), (2.27a,b). Here, in (2.27b) the set* $Y(\hat{x})$ *may be replaced by its subset* $Y_0(\hat{x})$.

(b) If, moreover, $q = 1$ *and the condition* $(\mathcal{TC})$ *holds, then we have:*
(2.7) for all $\xi \in I\!R^n$ *with (2.8), (2.27a,b), where* $Y_0(\hat{x})$ *is replaced by* $Y(\hat{x})$ *in (2.27b)*
$\Longrightarrow$ *(2.7) for all* $\xi \in I\!R^n$ *with (2.8), (2.9).*

Looking once again at our (standard) Example 2.8, then we see for those (special) situations that the above substitution of $Y(\hat{x})$ by $Y_0(\hat{x})$ and, hence, the subsequent problem modification, need not to be performed.

In order to formulate a further (general) sufficient optimality condition, let us modify the definition of $\mathcal{P}_{\mathcal{SI}}(f,h,g^\vee,v^\vee)$ a bit. Therefore, we introduce the following auxiliary feasible set and the corresponding auxiliary ordinary semi-infinite optimization problem:

$$
\begin{aligned}
M^{o,0}_{\mathcal{SI},\hat{x}}:&\quad \text{the feasible set } M^o_{\mathcal{SI}}[h,g^\vee] \ (K=\emptyset;\ \text{cf. (2.22)) up to replacing} \\
&\qquad\qquad Y(\hat{x}) \text{ by } Y_0(\hat{x}) \text{ in the definition of } Y^{\vee\,3,\ell} \text{ in (2.19a)} \\
\mathcal{P}^{o,0}_{\mathcal{SI},\hat{x}}:&\quad \text{Minimize } f(x) \text{ on } M^{o,0}_{\mathcal{SI},\hat{x}}
\end{aligned}
$$

Based on the considerations at the beginning of this section we learn from Lemma 2.9(b), that in the case of $q=1$ and under $(\mathcal{TC})$ the condition

$$\textit{(2.7)} \text{ for all } \xi \in I\!R^n \text{ with } \textit{(2.8), (2.9)}$$

is a necessary optimality condition at $\hat{x}$ with respect to $\mathcal{P}^{o,0}_{\mathcal{SI},\hat{x}}$. Then, however, we may by means of Theorem 2.4 state the following reversion of Lemma 2.5. This optimality criterion generalizes [28], Satz 4, Zusatz.

Corollary 2.10 (Theorem on a sufficient optimality condition (S2)). *Let for the generalized semi-infinite optimization problem* $\mathcal{P}_{\mathcal{SI}}(f,h,g,v)$ $(K=\emptyset)$ *a point* $\hat{x}\in I\!R^n$ *with* $\hat{x}\in M(\hat{x})$ *be given, and the Assumptions B, C, D, E, F hold (C-F referring to* $\hat{x}$*) or, locally on a neighborhood* $\mathcal{U}^0$ *of* $\hat{x}$*, the Assumptions* $B_{\mathcal{U}^0}$*-*$E_{\mathcal{U}^0}$*, F, be made. Moreover, let* $q=1$*, the condition* $(\mathcal{TC})$ *hold,* $\hat{x}$ *be a global or, say on* $\mathcal{U}^0$*, a local minimizer for* f *on* $M^{o,0}_{\mathcal{SI},\hat{x}}$*, LICQ for* $\hat{x}\in M[h]$ *and* $C^*_{\hat{x}}M^{o,0}_{\mathcal{SI},\hat{x}}\neq\emptyset$ *be fulfilled.*

Then, $\hat{x}\in M_{\mathcal{SI}}[h,g]$ *and* $\hat{x}$ *is a global or, with respect to the neighborhood* $\mathcal{U}^0$*, a local minimum for* $\mathcal{P}_{\mathcal{SI}}(f,h,g,v)$*, respectively.*

3. Concluding remark

In this paper we were concerned with some foundations of generalized semi-infinite optimization. Hereby, the relations with other investigations in literature were taken into consideration. In the modelling and in the results the local-global aspect was worked out. Based on two different approaches we arrived at representations of our generalized semi-infinite optimization problem by means of ordinary ones, and at both necessary and sufficient optimality conditions of first order. The two approaches were discussed, and a continued example was given.

This present investigation also serves as a preparation of numerical concepts for the purpose of solving our generalized semi-infinite optimization problem. Hereby, we have to realize and to use some properties on the topological behavior of the feasible sets which are involved (see [32]).

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