Вычислительные технологии, 2021, том 26, № 3, с. 86–106. © ФИЦ ИВТ, 2021 ISSN 1560-7534
Computational Technologies, 2021, vol. 26, no. 3, pp. 86–106. © FRC ICT, 2021 eISSN 2313-691X

COMPUTATIONAL TECHNOLOGIES

DOI:10.25743/ICT.2021.26.3.006

Stochastic optimization over the Pareto front by the augmented weighted Tchebychev program

Younsi-Abbaci L.*, Moulaï M.
USTHB University, LaROMaD laboratory, Faculty of Mathematics, BP 32 Bab-Ezzouar, El-Alia 16111, Algiers, Algeria
*Correspoding author: Leila Younsi-Abbaci, e-mail: abbaci.leila@yahoo.fr

Received December 14, 2020, revised April 21, 2021, accepted April 28, 2021.

In this paper, we propose a novel algorithm to deal with multi-objective stochastic integer linear programming problems (MOSILP). Given a stochastic linear function ϕ, we will optimize it over the full set of efficient solutions of a MOSILP. We convert the latter into an equivalent deterministic problem using uncertain aspirations which are inputs specified by the decision maker. For this purpose, we adopt a 2-stage recourse approach where an augmented weighted Tchebychev program is progressively optimized to generate an efficient solution, the value of the utility function ϕ is improved to enumerate all efficient solutions. The approach proposed here defines and solves a sequence of progressively more constrained integer linear programs, so that a new efficient solution is generated at each step of the algorithm. A numerical example is presented for illustration.

Keywords: multiple objective, integer programming, stochastic linear programming, Tchebychev norm.

Citation: Younsi-Abbaci L., Moulaï M. Stochastic optimization over the Pareto front by the augmented weighted Tchebychev program. Computational Technologies. 2021; 26(3):86–106. DOI:10.25743/ICT.2021.26.3.006.

Introduction

Let us consider the basic problem

$$(P_E)\begin{cases} \min & \Phi(x) = d(\xi)x, \\ \text{s.t.} & x \in \mathbf{E}_s, \end{cases} \tag{1}$$

where d is a random vector of dimension n and $\mathbf{E}_s$ is the efficient solution set of the multiple objective stochastic Integer linear programming problem MOSILP,

$$(MO_1)\begin{cases} \min & Z_i = C_i(\xi)x, \quad i = 1, ..., p, \\ \text{s.t.} & Ax = b, \\ & T(\xi)x = h(\xi), \\ & x \geq 0, \ \text{integer}, \end{cases} \tag{2}$$

where x is the decision variable vector of dimension $(n \times 1)$. C, T and h are random matrices of respective dimensions $(p \times n)$, $(m_1 \times n)$ and $(m_1 \times 1)$ with a joint probability distribution (independent again of the choice of x) defined on some probability space $(\Xi, E, prob)$. A and b are deterministic matrices of dimensions $(m \times n)$ and $(m \times 1)$, respectively. Let $C(\xi)$ be a $(p \times n)$random matrix with p rows $C_i(\xi) \in \mathbb{R}$.

The main difficulty of problem (P_E) arises from the nonconvexity of the efficient set $\mathbf{E}_s$, indeed $(\mathbf{E}_s)$ the union of several faces of the feasible set of problem (MO_1). Consequently, (P_E) is a global optimization problem.

(P_E) have been discussed extensively in the literature and a variety of methods have been developed for its solution (or resolution); see for example Philip in [1] studied the problem and described schematically a cutting plane procedure to solve it. Later, Isermann and Steuer [2] proposed a similar procedure for solving the problem they optimized one criterion among the multi-objective linear program functions. Necessary and sufficient conditions for this problem to be unbounded were established by Benson [3]. In [4], Ecker and Song used Philip's approach to introduce two implementable algorithms that involve a pivoting technique on the feasible set a reduced one of a multiple objectives integer linear program. Philip's method was implemented by Bolintineanu [5] for the case where the objective function of the problem is quasiconcave. Sayin in [6] formulated problem (P_E) as a linear program with an additional reserve convex constraint and proposed a cutting plane method to solve the latter problem. In [7], Abbas and Chaabane optimized linear function over an integer efficient set and Jorge developed in [8] another approach that defines a sequence of progressively more constrained single-objective integer problems that successively eliminates undesirable points, the most recent work on this topic was conducted by Chabaane *et al.* in [9].

The first interactive method for solving MOSILP problems was the STRANGE-MOMIX developed by Teghem [10]. In [11], Abbas and Belhacen (2006) proposed an algorithm that combines the cutting plane technique [12] and the L-shaped decomposition method described in [13]. The authors Amrouche and Moulaï (2012) developed in [14] an approach for detecting all stochastic integer efficient solutions of problem MOSILP based on solving a deterministic multiple objective integer linear program. When the decision variables are integers, few methods exist in the literature and cuts or branch and bound techniques are unavoidable.

In this paper, we propose an exact algorithm for solving (P_E), it is based on Jorge's approach [8] with the concepts L-shaped integer method [15]. We will use the Augmented Weighted Tchebychev program [16] to generate the set of nondominated objective vectors.

The remainder of the paper is organized as follows: in Section 1 we convert the problem MOSILP into an equivalent deterministic one; also, definitions and some results concerning the L-shaped decomposition method are given. Section 2 introduces the concepts of the utopian vector and the Augmented Weighted Tchebychev program. We describe our algorithm for optimizing a linear function over the efficient set of MOSILP in section 3. Every step of the method will be illustrated in Section 4 by a numerical example and Section 6 ends the paper with concluding remarks.

1. Construction of equivalent deterministic problem

The basic dual decomposition method for two-stage recourse problems is essentially an application of Benders decomposition [15], due to Van Slyke and Wets [13], and is usually called the L-shaped method in the literature. Assume that we have a joint finite discrete probability distribution $(\xi^r, prob^r)$, $r = 1, ..., R$, of the random data.

1. In the first stage, for each realization ξ^r of ξ, we associate a criterion $Z_{ir} = C_i(\xi^r)x$, a matrix $T(\xi^r)$ and a vector $h(\xi^r)$ to take into account the different scenarios affecting the p objectives and the stochastic constraints.
2. The second stage is to come back to the same idea of recourse used in single-criterion stochastic programming [17, 18]. Of course, we assume that the Decision Maker (DM) is able to specify the penalties $q^r = q(\xi^r)$ of the constraint violations y^r, $r = 1, ..., R$, and the size of the associated deterministic problem remains reasonable. Then, unlike the Strange method where a supplementary criterion is created to penalize the constraint violations, a recourse function $Q(x, \xi^r)$ is added to each criterion Z_{ir}. This penalty (called the recourse function) is given by:

$$Q(x, \xi^r) = \min_y \{(q^r)^T y | W(\xi^r) y = h(\xi^r) - T(\xi^r)x,\ y \geq 0\}. \tag{3}$$

Then the (DM) has to minimize the expected value of the total costs:

$$\widetilde{Z}_i = Esp[Z_i + Q(x, \xi)], \quad i = 1, \dots, p,$$

with Esp meaning expected value. It results in the following deterministic MOSILP problem

$$(MO_2) \begin{cases} \min & \widetilde{Z}_i = Z'_i + Q(x) \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \ \text{integer}, \end{cases} \tag{4}$$

where

$$Q(x) = Esp[Q(x, \xi)] = \sum_{r=1}^{R} prob^r (Q(x, \xi^r)) = \sum_{r=1}^{R} (prob^r q^r)^T y^r,$$

$$Z'_i = Esp[Z_i] = \sum_{i=1}^{p} prob^r C_i(\xi^r)x = Esp[C_i(\xi)x], \quad \text{note} \quad \widetilde{C}x = Esp[C_i(\xi)x]$$

are respectively the recourse-function $Q(x, \xi)$ and the expected values of $\widetilde{Z}_i$.

We expect the second-stage program (3) to be feasible for all the realizations ξ^r, $r = 1, ..., R$, of ξ. Depending on the $(m_1 \times n_1)$-recourse matrix $W(\xi^r)$, this needs not to be true for all the first-stage decisions $x \in \{x | Ax = b,\ x \geq 0\}$. Then, the first-stage decisions are restricted to $x \in \{x | Ax = b,\ x \geq 0\} \cap \mathbb{K} \neq \emptyset$ where, $\mathbb{K} = \{x | T(\xi^r)x + W(\xi^r)y^r = h(\xi^r),\ y^r \geq 0,\ r = 1, \dots, R\}$, is the induced first-stage feasibility set.

In the second-stage programs (3), the recourse-matrices $W(\xi^r)$, could be replaced by a fixed recourse-matrix W without any changes in the presentation of the proposed algorithm. Even if W is being fixed or not, the problem we face is that $\{x | Ax = b,\ x \geq 0\} \cap \mathbb{K}$ can be empty. To avoid this problem, complete fixed recourse-matrices that satisfy $\{t | t = Wy, t \geq 0\}$ are recommended. This implies that, whatever the first-stage decisions x and the realizations ξ^r of ξ turn out to be, the second-stage programs (3) are always feasible. A special case of complete fixed recourse matrix is simple recourse with the identity matrix I of order m_1, $W = (I, -I)$.

1.1. Associated relaxed problem and basic definitions

Associated to (P_E), the deterministic relaxed problem is defined by:

$$(P_R)\begin{cases} \min & \widetilde{\Phi}(x) = Esp(d(\xi)x) + Q(x) \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \text{ integer.} \end{cases} \tag{5}$$

1.2. Feasibility

The recourse-matrix W is being fixed. The question is then: how can we state that a given decision vector x^0 will yield feasible second-stage problems for all possible realisations of ξ. Therefore, it is a lot advantageous to work with the dual [17]

$$\max\{\pi^T(h(\xi) - T(\xi)x^0)|\, \pi W \leq q(\xi),\, \pi \in \mathbb{R}\} \tag{6}$$

on the other hand, the Farkas lemma states that $\{y|Wy = h(\xi) - T(\xi)x^0,\ y \geq 0\} \neq \emptyset$ if and only if $\sigma^T W \leq 0$ implies that $\sigma^T[h(\xi) - T(\xi)x^0] \leq 0$.

We conclude that $Q(x^0, \xi^r)$ is infeasible if and only if $P = \{\pi : \pi W \leq q(\xi)\}$ has an extreme ray σ such that $\sigma^T[h(\xi) - T(\xi)x^0] > 0$.

Then to check for feasibility of the second stage-problems, we have to find a direction vector σ by solving the dual problems:

$$\max\{\sigma^T(h(\xi) - T(\xi)x^0)\ \ \sigma^T W \leq 0,\, \|\sigma\|_1 \leq 1,\, \sigma \in \mathbb{R}\}, \tag{7}$$

where the constraint $\|\sigma\|_1 \leq 1$ is added to bound σ. In case where for some r, $r = 1, ..., R$ with r is the optimal solution of dual problem; we have $\sigma_r^T[h(\xi) - T(\xi)x^0] > 0$. Then we add the feasibility cut:

$$\sigma_r^T[h(\xi) - T(\xi)x^0] \leq 0. \tag{8}$$

1.3. Optimality

Assuming that all the feasibility cuts are there, we can reformulate the problem (5) by introducing a new variable θ:

$$\begin{cases} \min & \widetilde{\Phi}(x) = Esp(d(\xi)x) + \theta \\ \text{s.t.} & x \in D = \widetilde{D} \cap \mathbb{N}, \\ & \theta \geq Q(x), \\ & x \text{ integer,} \end{cases} \tag{9}$$

where $\widetilde{D} = \{x \in \mathbb{R}^n | Ax = b,\ \sigma_r^T(T(\xi^r) - h(\xi^r)) \geq 0,\ r = 1, ..., R\ \} = \{x \in \mathbb{R}^n | \widetilde{A}x = \widetilde{b}\}$. Throughout this paper, $\widetilde{D}$ is assumed to be a non-empty, compact polyhedron in $\mathbb{R}^n$.

The constraint

$$\theta \geq Q(x), \tag{10}$$

is in the optimality cut [17].

We define the notion of optimality for (MO_2) according to the Pareto concept.

Definition. *A point $x^* \in D$ is said to be efficient for (4) if and only if there does not exist another point $x^1 \in D$ such that $\widetilde{Z}_i(x^*) \geq \widetilde{Z}_i(x^1)$, $i \in 1, \ldots, p$, and $\widetilde{Z}_i(x^*) > \widetilde{Z}_i(x^1)$ for at least one $i \in 1, \ldots, p$ and for all the realisations, ξ^r $r = 1, \ldots, R$. Otherwise, x^* is not efficient and the corresponding vector $(\widetilde{Z}_1(x), \widetilde{Z}_2(x), \ldots, \widetilde{Z}_p(x))$ is said to be dominated.*

2. The utopian criterion vector and the Augmented Weighted Tchebychev program

Bowman [19] used a weighted Tchebychev norm for scalarization of multiple objective optimization problems. Based on this approach, Steuer and Choo [20] introduced the augmented weighted Tchebychev program and the lexicographic weighted Tchebychev program.

Let $\widetilde{Z}^{ideal} \in \mathbb{R}^p$ be the ideal criterion vector such that $\widetilde{Z}_i^{ideal} = \min\{\widetilde{Z}_i(x) | x \in D\}$. A vector strictly better than $\widetilde{Z}^{ideal}$ is called an utopian point $\widetilde{Z}^{utop}$, $\widetilde{Z}^{utop} < \widetilde{Z}^{ideal}$ or $\widetilde{Z}^{utop} = \widetilde{Z}^{ideal} - \vartheta$ where $\vartheta > 0$ and small. The augmented weighted Tchebychev norm of $\widetilde{Z}$ consist of measuring the distance between any criteria vector $\widetilde{Z}$ and the utopian vector $\widetilde{Z}^{utop}$, is defined as follows [19]:

$$\|\widetilde{Z}^{utop} - \widetilde{Z}\|_\infty^\lambda = \max_{i=1,\dots,p} \{\lambda_i |\widetilde{Z}_i^{utop} - \widetilde{Z}_i|\} + \rho \sum_{i=1}^p |\widetilde{Z}_i^{utop} - \widetilde{Z}_i|,$$

where ρ is a sufficiently small positive scalar and λ is weight vector.

Steuer (1986) has shown that if the ρ is small enough, the augmented weighted Tchebychev program not only guarantees to return a nondominated objective vector but generates any particular nondominated objective vector for an appropriate $\lambda \in \Lambda$. λ_i is the weight of the design objective i, and satisfies $\sum_{i=1}^p \lambda_i = 1$ and $\lambda_i \geq 0$, $i = 1, 2, ..., p$. The set

$$\Lambda = \left\{ \lambda \in \mathbb{R}^p | \sum_{i=1}^p \lambda_i = 1 \text{ and } \lambda_i \geq 0, \ \forall i \right\} \subset \mathbb{R}^p$$

is the weighting vector space and any $\lambda \in \Lambda$ is called a weighting vector.

The idea of this approach is to find a vector $\widetilde{Z}$ in the criteria space which minimizes the distance to the utopian vector:

$$\min_{Z \in \mathbf{Z}} \|\widetilde{Z}^{utop} - \widetilde{Z}\|_\infty^\lambda,$$

where $\mathbf{Z}$ is a feasible region in criteria space.

Theorem. [9] *Let $\lambda \in \Lambda$, for a small enough fixed $\rho > 0$, any optimal solution to $(P_\rho(\lambda))$ problem*

$$(P_\rho(\lambda)) \begin{cases} \min & \alpha + \rho \sum_{i=1}^p (\widetilde{Z}_i^{utop} - \widetilde{Z}_i) \\ \text{s.t.} & \alpha \geq \lambda_i (\widetilde{Z}_i^{utop} - \widetilde{Z}_i), \\ & x \in D, \\ & \alpha \geq 0 \end{cases} \tag{11}$$

is a nondominated objective vector to problem (MO_2). Here

$$\lambda_i = \frac{1}{\widetilde{Z}^{utop} - \widetilde{Z}_i^*} \left[\sum_{i=1}^p \frac{1}{\widetilde{Z}^{utop} - \widetilde{Z}_i^*} \right]^{-1} \quad \forall 1 \leq i \leq p,$$

with $\widetilde{Z}_i^* = C^i x^*$, where x^* is a prefixed vector in D.

3. Description of the method

Initially, the procedure determines the utopian objective vector $\widetilde{Z}^{utop}$. And the relaxed problem (P_R) associated to problem (P_E) is being solved, its feasible set is defined by deterministic constraints of problem (MO_1) without any feasibility or optimality cut. If for some realisations ξ^r, $r \in 1, \dots, R$, the second-stage problems yielded by the integer solution x found are not feasible, feasibility cuts (8) are introduced, we then get the integer optimal solution x, compute the recours function $Q(x)$. For a sufficiently small value ρ, the augmented weighted Tchebychev program $p_\rho(\lambda)$ is solved in order to find the nondominated vector $\bar{Z}$ that is the closest to the utopian objective vector $\widetilde{Z}^{utop}$ in the direction determined by Z^{utop} and $\bar{Z}$. We then get the integer optimal solution $\bar{x}$ of $p_\rho(\lambda)$, feasibility cuts (8) may be added if infeasibility of second-stage problems appears, and the corresponding value of θ.

Given that, in the decision space it may happen that the obtained solution is not better than an equivalent efficient solution on the main objective function , therefore, the following problem has to be solved to find an equivalent efficient solution which improves the main objective before reducing the current admissible region.

$$(T^l) : \min\{\widetilde{d}x | x \in D, \, \widetilde{C}x + \theta = \bar{Z}\}.$$

The optimal solution x^{*l} of this problem is considered as a first efficient solution.

Afterwards, at an iteration l, using Sylva and Crema's idea, see [21], we add to (P_R^l) new constraints that eliminate all the solutions dominated by x^{*l}. There by, the admissible domain is reduced. This task is performed by the resolution of the following problem P_R^l. It is worthnothing that all coefficients of $\widetilde{C}$ are supposed to be integers:

$$P_R^l \equiv \min\{\widetilde{d}x | x \in D - \cup_{s=1}^{l} D_s\}, \tag{12}$$

where $D_s = \{x \in \mathbb{Z}^n | \widetilde{C}x \geq \widetilde{C}x^s\}$ and $\{\widetilde{C}x^s\}_{s=1}^{l}$ is a subset of nondominated criteria vectors for problem (P_E). $\{x^s; s = 1, ..., l-1\}$ are solutions of (P_E) obtained at iterations $1, 2, ..., l-1$ respectively

$$\widetilde{H} = D - \cup_{s=1}^{l} D_s = \left\{ \begin{array}{l} \widetilde{C}^i x \leq (\widetilde{C}^i x^s + 1) y_i^s + M_i(1 - y_i^s), \\ i = 1, 2, ...p, s = 1, 2, ..., l, \\ \displaystyle\sum_{i=1}^{s} y_i^s \geq 1, \;\; s = 1, 2, ..., l, \\ y_i^s \in \{0, 1\}, i = 1, 2, ...p, s = 1, 2, ..., l, \\ x \in D \end{array} \right\},$$

where M_i is an upper bound for any feasible value of the i^{th} objective function. The associated variables y_i^s $i = 1, 2, ...p$, of x^{*s} and additional constraints are added to impose an improvement on at least one objective function. Note that when $y_i^s = 0$, the constraint is not restrictive, and when $y_i^s = 1$ a strict improvement is forced in the i^{th} objective function evaluated at x^{*s}.

3.1. Algorithm

The technical description of the method provides a new algorithm with an exponential complexity.

Algorithm: Stochastic optimizing over the Pareto optimal front

input :

$A_{(m\times n)}$: matrix of deterministic constraints
$b_{(m\times 1)}$: RHS vector;
$C_{(p\times n)}$: matrix criterion of stochastic coefficients;
$h(m_1 \times 1)$: vector of stochastic constraints ;
$W_{(m_1\times n_1)}$: matrix of stochastic constraints;
$T(m_1 \times n)$: matrix of stochastic constraints;

output :

x_{opt}: optimal solution of the problem (P_E), $\widetilde{\Phi}_{opt}$: optimal value of the main criterion $\widetilde{\Phi}$

initialization:

for $i \leftarrow 1$ **to** p **do**
  Solve $\widetilde{Z}_i^{ideal}= min\{\widetilde{C}^i x,x \in D\}$ is called the ideal point.
  and set the upper bounds $M_i = max\{\widetilde{C}^i x,x \in D\}$.

where $\vartheta = 1$ therefore $\widetilde{Z}^{utop} = \widetilde{Z}^{ideal} - 1$;
$\widetilde{\Phi}_{opt} := +inf$, $l := 1$, $E_1 := \emptyset$,
$Terminate := False$,, $\widetilde{H} := D$, $\theta := -\infty$;

while *Terminate:=False* **do**
  Solve $P_R \equiv min\{\widetilde{d}x|x \in \widetilde{H}\}$. Let x^l be an optimal solution of P_R;
  feasibility and optimality test
  for *r:=1 to R* **do**
    feasibility:=False;
    while *feasibility:=False* **do**
      $\hat{\sigma}^T$ an optimal solution of the problem:
      $max\{\sigma^T(h(\xi) - T(\xi)x^l) \mid \sigma^T W \leq 0, \ \|\sigma\|_1 \leq 1,\sigma \geq 0\}$
      coup:= $\hat{\sigma}^T(h(\xi) - T(\xi)x^l)$
      if *coup* > 0 **then**
        $\widetilde{H} = \widetilde{H} \bigcup \{\hat{\sigma}^T[h(\xi) - T(\xi)x) < 0\}$;
        Let x^l be an optimal solution of R_2^l
      else
        Fallibility:= true
  recours function Q:=0;
  for *r:=1 to R* **do**
    solve problem: $\max\{\pi^T(h(\xi) - T(\xi)x^l) \mid \pi W \leq q(\xi)\}$
    $Q(x) := Q(x) + prob^r \times Q(x^l, \xi^r)$
  $\theta := Q(x)$,
  (x^l, θ): optimal solution for P_R^l (after feasibility and optimality tests)
  if *Terminate:=false or* $\widetilde{\Phi}(x^l) \geq \widetilde{\Phi}_{opt}$ **then**
    x_{opt} an optimal solution of P_E, **Terminate:=True**.
  else
    efficiency test of $\widetilde{Z}^l = \widetilde{C}x_T^l$;
    compute the weighted vector of $P_\rho(\lambda^l)$;
    Let $(\bar{x}^l, \bar{Z}^l)$ be an optimal solution of $P_\rho(\lambda^l)$
    feasibility and optimality for $\bar{x}^l$
    if $\widetilde{d}\bar{x}^l = \widetilde{\Phi}_{opt}$ **then**
      $x_{opt} = \bar{x}^l$, $\widetilde{\Phi}_{opt} = \widetilde{\Phi}(\bar{x}^l)$, **Terminated:=True**
    else
      solve $T(\bar{x}^l) = min\{\widetilde{d}x + \theta|x \in D, \widetilde{C}x + \theta = \bar{Z}^l\}$;
      let x^{*l} be an optimal solution of $T(\bar{x}^l)$;
      feasibility and optimality test if $\widetilde{\Phi}_{opt} \leq \widetilde{\Phi}(x^{*l})$ **then**
        x_{opt} an optimal solution of P_E, **Terminate:=True**
      else
        $x_{opt} := x^{*l}$, $\widetilde{\Phi}_{opt} := \widetilde{\Phi}(x^{*l})$; let $\mathbf{E}_s^{l+1} = \mathbf{E}_s^l \cup x^{*l}$;
        $l := l + 1$ and $\widetilde{H} := D \backslash \cup D_{s=1}^{l-1}$;

Proposition. *The algorithm terminates in a finite number of iterations.*

Proof. Since there are finitely many feasible bases coming from the recourse-matrix W, there are only finitely many feasibility and optimality cuts. On other hand, at each iteration of the algorithm, a new improved efficient solution is generated and the admissible region is being reduced there until infeasibility. All these additional cuts exclude the points or the edges once scanned, leading to the convergence of the procedure in a finite number of steps. □

4. An illustrative example

The problem of optimization over the efficient set of the MOSILP.

Two scenarios ($R = 2$).

- *The principal problem*:

$$d(\xi^1) = (4, -10), \quad d(\xi^2) = (-6, 12),$$

we calculate the expected value $\widetilde{d}$

$$\begin{aligned}\widetilde{d} = Esp(d(x, \xi)) &= \frac{1}{2}d(\xi^1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{1}{2}d(\xi^2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \\ &= \frac{1}{2}(4, -10)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{1}{2}(-6, 12)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \\ &= -x_1 + x_2\end{aligned}$$

and then we obtain a linear function optimization problem over an efficient set

$$(P_E)\begin{cases} \min & \Phi(x) = -x_1 + x_2 \\ \text{s.t.} & x \in \mathbf{E}_s. \end{cases} \tag{13}$$

- *Multiobjective stochastic problem*: let us consider the following example with a structure similar to problem (MO_1), $p = 2$, $n_1 = 4$, $m_1 = m = n = 2$.
  - **Matrix C:**

$$C_1(\xi^1) = (4, -9), \quad C_1(\xi^2) = (-6, 3),$$
$$C_2(\xi^1) = (8, 5), \quad C_2(\xi^2) = (-2, -3).$$

  - **Matrix T and vector h:**

$$\mathbf{T}(\xi^1) = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad \mathbf{T}(\xi^2) = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix},$$

$$\mathbf{h}(\xi^1) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \quad \mathbf{h}(\xi^2) = \begin{pmatrix} 6 \\ 1 \end{pmatrix}.$$

  - **The penalties:**

$$q(\xi^1) = \begin{pmatrix} 1 & 0 & 6 & 2 \end{pmatrix}^T, \quad q(\xi^2) = \begin{pmatrix} 5 & 3 & 2 & 1 \end{pmatrix}^T,$$

$$prob(\xi^1) = \frac{1}{2}, \quad prob(\xi^2) = \frac{1}{2}.$$

  - **Recourse-matrix:**

$$\mathbf{W}(\xi) = \mathbf{W} = \begin{pmatrix} -2 & -1 & 2 & 1 \\ 3 & 2 & -5 & -6 \end{pmatrix},$$

we calculate the expected value $\bar{Z}_1$ and $\bar{Z}_2$:

$$\begin{aligned}
\bar{Z}_1 = Esp(Z_1(x,\xi)) &= \frac{1}{2}C_1(\xi^1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{1}{2}C_1(\xi^2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \\
&= \frac{1}{2}(4,-9)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{1}{2}(-6,3)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \\
&= -x_1 - 3x_2,
\end{aligned}$$

$$\begin{aligned}
\bar{Z}_2 = Esp(Z_2(x,\xi)) &= \frac{1}{2}C_2(\xi^1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{1}{2}C_2(\xi^2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \\
&= \frac{1}{2}(8,5)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{1}{2}(-2,-3)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \\
&= 3x_1 + x_2.
\end{aligned}$$

Deterministic constraints

$$\begin{aligned}
-2x_1 + 5x_2 &\le 23, \\
8x_1 + x_2 &\le 55, \\
x_1 - x_2 &\le 4, \\
-x_1 - x_2 &\le -6.
\end{aligned}$$

Stochastic constraints

$$\text{First scenario } \xi^1 \quad \begin{aligned} x_1 + 2x_2 &= 3, \\ -2x_1 + x_2 &= 5. \end{aligned}$$

$$\text{Second scenario } \xi^2 \quad \begin{aligned} x_1 + 0x_2 &= 6, \\ 3x_1 + 4x_2 &= 1. \end{aligned}$$

We obtain the following deterministic multiple objective integer linear programming problem:

$$MO_1 \begin{cases} \min\ \widetilde{Z}_1 = -x_1 - 3x_2 + Q(x), \\ \min\ \widetilde{Z}_2 = 3x_1 + x_2 + Q(x) \\ \text{s.t. } D = \begin{cases} -2x_1 + 5x_2 \le 23, \\ 8x_1 + x_2 \le 55, \\ x_1 - x_2 \le 4, \\ -x_1 - x_2 \le -6, \\ x_1, x_2 \ge 0, \ \text{integer}, \end{cases} \end{cases} \tag{14}$$

with $Q(x) = \frac{1}{2}Q(x,\xi^1) + \frac{1}{2}Q(x,\xi^2)$, and the second stage problem associated with the two scenarios $Q(x,\xi^1)$ and $Q(x,\xi^2)$ respectively:

$$\mathbf{Q}(x,\xi^1) \begin{cases} \min & y_1 + 6y_3 + 2y_4 \\ \text{s.t.} & -2y_1 - y_2 + 2y_3 + y_4 = 3 - x_1 - 2x_2, \\ & 3y_1 + 2y_2 - 5y_3 - 6y_4 = 5 + 2x_1 - x_2, \\ & y \ge 0, \end{cases} \tag{15}$$

$$\mathbf{Q}(x,\xi^2) \begin{cases} \min & 5y_1 + 3y_2 + 2y_3 + y_4 \\ \text{s.t.} & -2y_1 - y_2 + 2y_3 + y_4 = 6 - x_1, \\ & 3y_1 + 2y_2 - 5y_3 - 6y_4 = 1 + 3x_1 - 4x_2, \\ & y \ge 0. \end{cases} \tag{16}$$

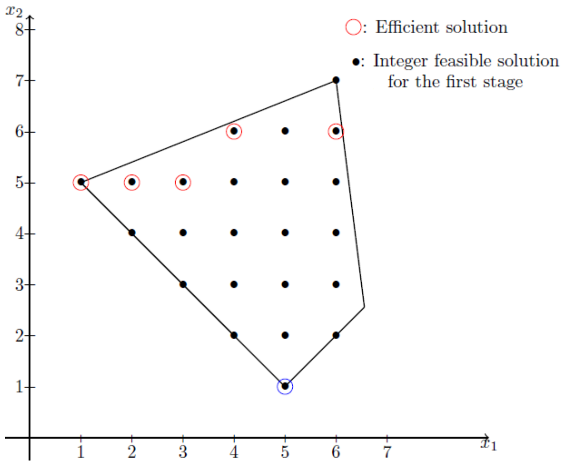

Fig. 1. Admissibility domain D^1 with efficient set

We can use one of the algorithms developed in [11] to find the efficient set, it can be shown that five of them are efficient. Particularly, the efficient set $\mathbf{E}_s$ is given by

$$\mathbf{E}_s = \{(1,5);(2,5);(3,5);(4,6);(6,6)\},$$

as shown in Fig. 1.

For this example, the parameter ρ has been fixed at 0.001.

Initial iteration

- We calculate the upper bound of each objective function $M_1 = -7.25$, $M_2 = 36.9855$, $D^1 = D$, $\theta = -\infty$;
- $\widetilde{Z}_1^{ideal} = -15.765$, $\widetilde{Z}_2^{ideal} = 14$, $\widetilde{Z}_1^{utop} = -16.765$, $\widetilde{Z}_2^{utop} = 13$.
- **Step 1.** With $\theta = -\infty$ and without feasibility and optimality cuts, solve the main deterministic relaxed problem P_R under the deterministic constraints $P_R^1 \equiv \left\{ \min \quad \Phi(x) = -x_1 + x_2 | x \in D \right\}$, an optimal solution is $x^1 = (5,1)$. To test the feasibility of the second-stage problems (15) and (16), we solve the program (7) with:

$$h(\xi^1) - T(\xi^1)x^1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 14 \end{pmatrix},$$
$$h(\xi^2) - T(\xi^2)x^1 = \begin{pmatrix} 6 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -18 \end{pmatrix}.$$

$\begin{array}{ll} \max & -4\sigma_1^1 + 14\sigma_1^2 \\ \text{s.t.} & -2\sigma_1^1 + 3\sigma_1^2 \le 0, \\ & -1\sigma_1^1 + 2\sigma_1^2 \le 0, \\ & 2\sigma_1^1 - 5\sigma_1^2 \le 0, \\ & 1\sigma_1^1 - 6\sigma_1^2 \le 0, \\ & 1\sigma_1^1 + 1\sigma_1^2 \le 1 \end{array}$	maximum is at $\sigma_1^T = (\sigma_1^1, \sigma_1^2) = \left(\frac{2}{3}, \frac{1}{3}\right)$
$\begin{array}{ll} \max & 1\sigma_2^1 + -18\sigma_2^2 \\ \text{s.t.} & -2\sigma_2^1 + 3\sigma_2^2 \le 0, \\ & -1\sigma_2^1 + 2\sigma_2^2 \le 0, \\ & 2\sigma_2^1 - 5\sigma_2^2 \le 0, \\ & 1\sigma_2^1 - 6\sigma_2^2 \le 0, \\ & 1\sigma_2^1 + 1\sigma_2^2 \le 1 \end{array}$	maximum is at $\sigma_2^T = (\sigma_2^1, \sigma_2^2) = (0,0)$

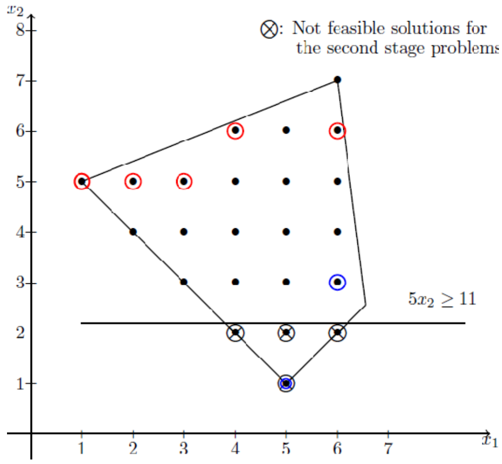

Fig. 2. Admissibility domain without stochastic constraint

$$\sigma_1^T[h(\xi^1) - T(\xi^1)x^1] = \left(\frac{2}{3}, \frac{1}{3}\right)\begin{pmatrix} -4 \\ 14 \end{pmatrix} = 2 > 0,$$

$$\sigma_2^T[h(\xi^2) - T(\xi^2)x^1] = \left(\frac{5}{7}, \frac{2}{7}\right)\begin{pmatrix} 1 \\ -18 \end{pmatrix} = 0.$$

$\sigma_1^T[h(\xi^1) - T(\xi^1)x^1] > 0$. It means that the second-stage is not feasible for the ξ^1. Then we create a feasibility cut

$$\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix} \iff 5x_1 \geq 11.$$

The cut is added to the first problem P_R^1. We get a new integer point $(6, 3)$ (see Fig. 2).

To test the feasibility of the second-stage problems (15) and (16), we solve the program (7) with:

$$h(\xi^1) - T(\xi^1)(6,3) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} -9 \\ 14 \end{pmatrix},$$

$$h(\xi^2) - T(\xi^2)(6,3) = \begin{pmatrix} 6 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -29 \end{pmatrix}.$$

max	$-9\sigma_1^1 + 14\sigma_1^2$	
T.Q.	$-2\sigma_1^1 + 3\sigma_1^2 \leq 0,$	
	$-1\sigma_1^1 + 2\sigma_1^2 \leq 0,$	maximum is at
	$2\sigma_1^1 - 5\sigma_1^2 \leq 0,$	$\sigma_1^T = (\sigma_1^1, \sigma_1^2) = (0, 0)$
	$1\sigma_1^1 - 6\sigma_1^2 \leq 0,$	
	$1\sigma_1^1 + 1\sigma_1^2 \leq 1$	

max	$0\sigma_2^1 - 29\sigma_2^2$	
T.Q.	$-2\sigma_2^1 + 3\sigma_2^2 \leq 0,$	
	$-1\sigma_2^1 + 2\sigma_2^2 \leq 0,$	maximum is at
	$2\sigma_2^1 - 5\sigma_2^2 \leq 0,$	$\sigma_2^T = (\sigma_2^1, \sigma_2^2) = (0, 0)$
	$1\sigma_2^1 - 6\sigma_2^2 \leq 0,$	
	$1\sigma_2^1 + 1\sigma_2^2 \leq 1$	

$$\sigma_1^T[h(\xi^1) - T(\xi^1)(6,3)] = (0,0)\begin{pmatrix} -9 \\ 14 \end{pmatrix} = 0,$$

$$\sigma_2^T[h(\xi^2) - T(\xi^2)(6,3)] = (0,0)\begin{pmatrix} 0 \\ -29 \end{pmatrix} = 0,$$

$\sigma_1 = \sigma_2 = 0$, this implies that the solution $x^1 = (6,3)$ yields feasible second-stage problems. To test the optimality of $x^1 = (6,3)$, the dual (6) is solved for ξ^1 and ξ^2.

max	$-9\pi_1^1 + 14\pi_1^2$	maximum is at $\pi_1^T = (\pi_1^1, \pi_1^2) = \left(-1, -\frac{1}{2}\right)$
T.Q.	$-2\pi_1^1 + 3\pi_1^2 \leq 1,$	
	$-1\pi_1^1 + 2\pi_1^2 \leq 0,$	
	$2\pi_1^1 - 5\pi_1^2 \leq 6,$	
	$1\pi_1^1 - 6\pi_1^2 \leq 2$	

max	$0\pi_2^1 - 29\pi_2^2$	maximum is at $\pi_2^T = (\pi_2^1, \pi_2^2) = (0, -0.17)$
T.Q.	$-2\pi_2^1 + 3\pi_2^2 \leq 5,$	
	$-1\pi_2^1 + 2\pi_2^2 \leq 3,$	
	$2\pi_2^1 - 5\pi_2^2 \leq 2,$	
	$1\pi_2^1 - 6\pi_2^2 \leq 1$	

We calculate the value of the second stage problem associated with the two scenarios $Q(x^1, \xi^1)$ and $Q(x^1, \xi^2)$ respectively:

$$\mathbf{Q}(x^1, \xi^1) = \pi_1^T[h(\xi^1) - T(\xi^1)x^1] = \left(-1, -\frac{1}{2}\right)\begin{pmatrix} -9 \\ 14 \end{pmatrix} = 2,$$

$$\mathbf{Q}(x^1, \xi^1) = \pi_2^T[h(\xi^2) - T(\xi^2)x^1] = (0, -0.17)\begin{pmatrix} 0 \\ -29 \end{pmatrix} = 4.93,$$

$$\mathbf{Q}(x^1) = \frac{1}{2}\mathbf{Q}(x^1, \xi^1) + \frac{1}{2}\mathbf{Q}(x^1, \xi^1) = \frac{1}{2}(2 + 4.93) = 3.465.$$

$\theta = -\infty < \mathbf{Q}(x^1)$ we introduce the optimality cut of the form

$$\theta > \sum_{r=1}^{R} p^r[h(\xi_r) - T(\xi_r)x],$$

adding this cut $\theta \geq 2.835 + 0.255x_1 + 1.593x_2$, and reoptimize the precedent program P_R^1, $\theta = 15.315$, then $x^1 = (6,3)$ is the optimal basic feasible solution on the current region. Let $\widetilde{Z}^1(x^1) = (-15, 21) + 3.465 = (-11.535, 24.465)$, $\widetilde{\Phi}(6,3) = -3 + \theta = -0.465$, $\widetilde{\Phi}(6,3) < \widetilde{\Phi}_{opt}$. We compute the weighted vector λ^1 of the $\widetilde{Z}^1$:

$$\lambda_1^1 = \frac{1}{-16.765 - (-11.535)}\left[\frac{1}{(-16.765) - (-11.535)} + \frac{1}{(13) - (24.465)}\right]^{-1},$$

$$\lambda_2^1 = \frac{1}{(13) - (24.465)}\left[\frac{1}{(-16.765) - (-11.535)} + \frac{1}{(13) - (24.465)}\right]^{-1},$$

$$\lambda^1 = (0.686, 0.314).$$

- **Step 2.** We solve the generalized Tchebychev program $(P_\rho(\lambda^1))$, which is defined as follows:

$$(P_\rho(\lambda^1))\begin{cases} \min & \alpha + 0.001(-3.7765 - 2x_1 + 2x_2) \\ \text{s.t.} & \alpha \geq 0.686(-16.765 + x_1 + 3x_2), \\ & \alpha \geq 0.313(13 - 3x_1 - x_2), \\ & x \in D. \end{cases} \tag{17}$$

Then, $\bar{x}^1 = (2,5)$ is the optimal basic feasible solution of $(P_\rho(\lambda^1))$. To test the feasibility of the second-stage problems (15) and (16), we solve the program (7) with:

$$h(\xi^1) - T(\xi^1)\bar{x}^1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -9 \\ 4 \end{pmatrix},$$

$$h(\xi^2) - T(\xi^2)\bar{x}^1 = \begin{pmatrix} 6 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ -25 \end{pmatrix}.$$

max $-9\sigma_1^1 + 4\sigma_1^2$ T.Q. $-2\sigma_1^1 + 3\sigma_1^2 \leq 0,$ $-1\sigma_1^1 + 2\sigma_1^2 \leq 0,$ $2\sigma_1^1 - 5\sigma_1^2 \leq 0,$ $1\sigma_1^1 - 6\sigma_1^2 \leq 0,$ $1\sigma_1^1 + 1\sigma_1^2 \leq 1$	maximum is at $\sigma_1^T = (\sigma_1^1, \sigma_1^2) = (0,0)$
max $4\sigma_2^1 - 25\sigma_2^2$ T.Q. $-2\sigma_2^1 + 3\sigma_2^2 \leq 0,$ $-1\sigma_2^1 + 2\sigma_2^2 \leq 0,$ $2\sigma_2^1 - 5\sigma_2^2 \leq 0,$ $1\sigma_2^1 - 6\sigma_2^2 \leq 0,$ $1\sigma_2^1 + 1\sigma_2^2 \leq 1$	maximum is at $\sigma_2^T = (\sigma_2^1, \sigma_2^2) = (0,0)$

The solution is feasible for both first and second scenario.
To test the optimality of $\bar{x}^1 = (2,5)$, the dual (6) is solved for ξ^1 and ξ^2.

max $-9\pi_1^1 + 4\pi_1^2$ T.Q. $-2\pi_1^1 + 3\pi_1^2 \leq 1,$ $-1\pi_1^1 + 2\pi_1^2 \leq 0,$ $2\pi_1^1 - 5\pi_1^2 \leq 6,$ $1\pi_1^1 - 6\pi_1^2 \leq 2$	maximum is at $\pi_1^T = (\pi_1^1, \pi_1^2) = \left(-1, -\frac{1}{2}\right)$
max $4\pi_2^1 - 25\pi_2^2$ T.Q. $-2\pi_2^1 + 3\pi_2^2 \leq 5,$ $-1\pi_2^1 + 2\pi_2^2 \leq 3,$ $2\pi_2^1 - 5\pi_2^2 \leq 2,$ $1\pi_2^1 - 6\pi_2^2 \leq 1$	maximum is at $\pi_2^T = (\pi_2^1, \pi_2^2) = (0, -0.17)$

We calculate the value of the second stage problem associated with the two scenarios $Q(\bar{x}^1, \xi^1)$ and $Q(\bar{x}^1, \xi^2)$ respectively:

$$\mathbf{Q}(\bar{x}^1, \xi^1) = \pi_1^T[h(\xi^1) - T(\xi^1)\bar{x}^1] = \left(-1, -\frac{1}{2}\right)\begin{pmatrix} -9 \\ 4 \end{pmatrix} = 7,$$

$$\mathbf{Q}(\bar{x}^1, \xi^1) = \pi_2^T[h(\xi^2) - T(\xi^2)\bar{x}^1] = (0, -0.17)\begin{pmatrix} 4 \\ -25 \end{pmatrix} = 4.25,$$

$$\mathbf{Q}(\bar{x}^1) = \frac{1}{2}\mathbf{Q}(\bar{x}^1, \xi^1) + \frac{1}{2}\mathbf{Q}(\bar{x}^1, \xi^1) = 5.625.$$

The solution $\widetilde{Z}^1(\bar{x}^1) = (-17, 11) + 5.625 = (-11.375, 16.625)$ is a nondominated point with minimal weighted Tchebychev distance, we obtain $\bar{x}^1 = (2, 5)$ and $\widetilde{\Phi}(2, 5) = 3 + 5.625 = 8.625$.

- **Step 3.** Solve the equivalent efficient solutions program:

$$(T^1)\begin{cases} \min & \Phi(x) = -x_1 + x_2 + \theta \\ \text{s.t.} & x_1, x_2 \in D, \\ & -x_1 - 3x_2 + \theta = -11.375, \\ & 3x_1 + 1x_2 + \theta = 16.625. \end{cases} \tag{18}$$

An optimal solution is $x^{*1} = x^1 = (2, 5)$ with $\theta = 5.625$, $\widetilde{\Phi}(x^{*1}) = 8.625 < \widetilde{\Phi}_{opt}$, $x_{opt} := x^{*1}$, $\widetilde{\Phi}_{opt} := \widetilde{d}x^{*1}$, and let $\mathbf{E}_s^1 = \{(2, 5)\}$, $l := l + 1 = 2$ and we solve problem P_R^2.

Iteration 2

- **Step 1.**

$$(P_R^2)\begin{cases} \min & \widetilde{\Phi}(x) = -x_1 + x_2 + \theta \\ \text{s.t.} & x_1, x_2 \in \tilde{H}, \\ & -x_1 - 3x_2 + \theta \le (-11.375 + 1)y_1^1 - 7.25(1 - y_1^1), \\ & 3x_1 + x_2 + \theta \le (16.625 + 1)y_2^1 + 36.985(1 - y_2^1), \\ & y_1^1 + y_1^1 \ge 1, \quad y_1^1, y_2^1 \in \{0, 1\}. \end{cases} \tag{19}$$

An optimal solution is $x^2 = (6, 4)$, with $y^1 = (1, 0)$. To test the feasibility of the second-stage problems (15) and (16), we solve the program (7) with:

$$h(\xi^1) - T(\xi^1)x^2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} -11 \\ 13 \end{pmatrix},$$

$$h(\xi^2) - T(\xi^2)x^2 = \begin{pmatrix} 6 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -33 \end{pmatrix}.$$

max	$-11\sigma_1^1 + 13\sigma_1^2$	
T.Q.	$-2\sigma_1^1 + 3\sigma_1^2 \le 0,$	
	$-1\sigma_1^1 + 2\sigma_1^2 \le 0,$	maximum is at
	$2\sigma_1^1 - 5\sigma_1^2 \le 0,$	$\sigma_1^T = (\sigma_1^1, \sigma_1^2) = (0, 0)$
	$1\sigma_1^1 - 6\sigma_1^2 \le 0,$	
	$1\sigma_1^1 + 1\sigma_1^2 \le 1$	
max	$0\sigma_2^1 - 33\sigma_2^2$	
T.Q.	$-2\sigma_2^1 + 3\sigma_2^2 \le 0,$	
	$-1\sigma_2^1 + 2\sigma_2^2 \le 0,$	maximum is at
	$2\sigma_2^1 - 5\sigma_2^2 \le 0,$	$\sigma_2^T = (\sigma_2^1, \sigma_2^2) = (0, 0)$
	$1\sigma_2^1 - 6\sigma_2^2 \le 0,$	
	$1\sigma_2^1 + 1\sigma_2^2 \le 1$	

The solution is feasible for both first and second scenario. To test the optimality of $x^2 = (6, 4)$, the dual (6) is solved for ξ^1 and ξ^2.

max $-11\pi_1^1 + 13\pi_1^2$ T.Q. $-2\pi_1^1 + 3\pi_1^2 \leq 1,$ $-1\pi_1^1 + 2\pi_1^2 \leq 0,$ $2\pi_1^1 - 5\pi_1^2 \leq 6,$ $1\pi_1^1 - 6\pi_1^2 \leq 2$	maximum is at $\pi_1^T = (\pi_1^1, \pi_1^2) = \left(-1, -\frac{1}{2}\right)$

max $0\pi_2^1 - 33\pi_2^2$ T.Q. $-2\pi_2^1 + 3\pi_2^2 \leq 5,$ $-1\pi_2^1 + 2\pi_2^2 \leq 3,$ $2\pi_2^1 - 5\pi_2^2 \leq 2,$ $1\pi_2^1 - 6\pi_2^2 \leq 1$	maximum is at $\pi_2^T = (\pi_2^1, \pi_2^2) = (0, -0.17)$

We calculate the value of the second stage problem associated with the two scenarios $Q(x^2, \xi^1)$ and $Q(x^2, \xi^2)$ respectively:

$$\mathbf{Q}(x^2, \xi^2) = \pi_1^T[h(\xi^1) - T(\xi^1)x^2] = \left(-1, -\frac{1}{2}\right)\begin{pmatrix} -11 \\ 13 \end{pmatrix} = 4.5,$$

$$\mathbf{Q}(x^2, \xi^1) = \pi_2^T[h(\xi^2) - T(\xi^2)x^2] = (0, -0.17)\begin{pmatrix} 0 \\ -33 \end{pmatrix} = 5.2,$$

$$\mathbf{Q}(x^2) = \frac{1}{2}\mathbf{Q}(x^2, \xi^1) + \frac{1}{2}\mathbf{Q}(x^2, \xi^1) = 5.1.$$

Then $x^2 = (6, 4)$ is the optimal basic feasible solution in the current region. Let $\widetilde{Z}^2(x^2) = (-18, 22) + 5.1 = (-12.9, 27.3)$, $\widetilde{\Phi}(6, 4) = -2 + \theta = 3.1$, $\widetilde{\Phi}(6, 4) < \widetilde{\Phi}_{opt}$. We compute the weighted vector λ^2 of the $\widetilde{Z}^2$: $\lambda^2 = (0.782, 0.218)$.

- **Step 2.** We solve the generalized Tchebychev program $(P_\rho(\lambda^2))$, which is defined as follows:

$$(P_\rho(\lambda^2)) \begin{cases} \min & \alpha + 0.001(-3.7765 - 2x_1 + 2x_2) \\ \text{s.t.} & \alpha \geq 0.782(-16.765 + x_1 + 3x_2), \\ & \alpha \geq 0.218(13 - 3x_1 - x_2), \\ & x \in D. \end{cases} \tag{20}$$

Then, $\bar{x}^2 = (3, 5)$ is the optimal basic feasible solution of $(P_\rho(\lambda^2))$. To test the feasibility of the second-stage problems (15) and (16), we solve the program (7) with:

$$h(\xi^1) - T(\xi^1)\bar{x}^2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -10 \\ 6 \end{pmatrix},$$

$$h(\xi^2) - T(\xi^2)\bar{x}^2 = \begin{pmatrix} 6 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -28 \end{pmatrix}.$$

max $-10\sigma_1^1 + 6\sigma_1^2$ T.Q. $-2\sigma_1^1 + 3\sigma_1^2 \leq 0,$ $-1\sigma_1^1 + 2\sigma_1^2 \leq 0,$ $2\sigma_1^1 - 5\sigma_1^2 \leq 0,$ $1\sigma_1^1 - 6\sigma_1^2 \leq 0,$ $1\sigma_1^1 + 1\sigma_1^2 \leq 1$	maximum is at $\sigma_1^T = (\sigma_1^1, \sigma_1^2) = (0, 0)$

max $3\sigma_2^1 - 28\sigma_2^2$ T.Q. $-2\sigma_2^1 + 3\sigma_2^2 \leq 0,$ $-1\sigma_2^1 + 2\sigma_2^2 \leq 0,$ $2\sigma_2^1 - 5\sigma_2^2 \leq 0,$ $1\sigma_2^1 - 6\sigma_2^2 \leq 0,$ $1\sigma_2^1 + 1\sigma_2^2 \leq 1$	maximum is at $\sigma_2^T = (\sigma_2^1, \sigma_2^2) = (0, 0)$

The solution is feasible for both first and second scenario. To test the optimality of $\bar{x}^2 = (3,5)$, the dual (6) is solved for ξ^1 and ξ^2.

$\begin{array}{ll} \max & -10\pi_1^1 + 6\pi_1^2 \\ \text{T.Q.} & -2\pi_1^1 + 3\pi_1^2 \leq 1, \\ & -1\pi_1^1 + 2\pi_1^2 \leq 0, \\ & 2\pi_1^1 - 5\pi_1^2 \leq 6, \\ & 1\pi_1^1 - 6\pi_1^2 \leq 2 \end{array}$	maximum is at $\pi_1^T = (\pi_1^1, \pi_1^2) = \left(-1, -\frac{1}{2}\right)$
$\begin{array}{ll} \max & 3\pi_2^1 - 28\pi_2^2 \\ \text{T.Q.} & -2\pi_2^1 + 3\pi_2^2 \leq 5, \\ & -1\pi_2^1 + 2\pi_2^2 \leq 3, \\ & 2\pi_2^1 - 5\pi_2^2 \leq 2, \\ & 1\pi_2^1 - 6\pi_2^2 \leq 1 \end{array}$	maximum is at $\pi_2^T = (\pi_2^1, \pi_2^2) = (0, -0.17)$

We calculate the value of the second stage problem associated with the two scenarios $Q(\bar{x}^2, \xi^1)$ and $Q(\bar{x}^2, \xi^2)$ respectively:

$$\mathbf{Q}(\bar{x}^2, \xi^1) = \pi_1^T[h(\xi^1) - T(\xi^1)\bar{x}^2] = \left(-1, -\frac{1}{2}\right)\begin{pmatrix} -10 \\ 6 \end{pmatrix} = 7,$$

$$\mathbf{Q}(\bar{x}^2, \xi^1) = \pi_2^T[h(\xi^2) - T(\xi^2)\bar{x}^2] = (0, -0.17)\begin{pmatrix} 3 \\ -28 \end{pmatrix} = 4.76,$$

$$\mathbf{Q}(\bar{x}^2) = \frac{1}{2}\mathbf{Q}(\bar{x}^2, \xi^1) + \frac{1}{2}\mathbf{Q}(\bar{x}^2, \xi^1) = 5.88.$$

The solution $\widetilde{Z}^2(\bar{x}^2) = (-18, 14) + 5.88 = (-13.88, 19.88)$ is a nondominated point with minimal weighted Tchebychev distance, we obtain $\bar{x}^2 = (3,5)$ and $\widetilde{\Phi}(3,5) = 2 + 5.88 = 7.88$.

- **Step 3.** Solve the equivalent efficient solutions program:

$$(T^2)\begin{cases} \min & \Phi(x) = -x_1 + x_2 + \theta \\ \text{s.t.} & x_1, x_2 \in D, \\ & -x_1 - 3x_2 + \theta = -13.88, \\ & 3x_1 + 1x_2 + \theta = 19.88. \end{cases} \tag{21}$$

An optimal solution is $x^{*2} = x^2 = (3,5)$ with $\theta = 5.88$, $\widetilde{\Phi}(x^{*2}) = 7.88 < \widetilde{\Phi}_{opt}$, $x_{opt} := x^{*2}$, $\widetilde{\Phi}_{opt} := \widetilde{d}x^{*2}$, and let $\mathbf{E}_s^2 = \{(2,5),(3,5)\}$, $l := l + 2 = 3$ and we solve problem (P_R^3).

Iteration 3

- **Step 1.**

$$(P_R^3)\begin{cases} \min & \widetilde{\Phi}(x) = -x_1 + x_2 + \theta \\ \text{s.t.} & x_1, x_2 \in \tilde{H}, \\ & -x_1 - 3x_2 + \theta \leq (-13.88 + 1)y_1^1 - 7.25(1 - y_1^1), \\ & 3x_1 + x_2 + \theta \leq (19.88 + 1)y_2^1 + 36.985(1 - y_2^1), \\ & y_1^1 + y_1^1 \geq 1, \quad y_1^1, y_2^1 \in \{0,1\}. \end{cases} \tag{22}$$

An optimal solution is $x^3 = (6,5)$, with $y^1 = (1,0)$. To test the feasibility of the second-stage problems (15) and (16), we solve the program (7) with:

$$h(\xi^1) - T(\xi^1)x^3 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} -13 \\ 12 \end{pmatrix},$$

$$h(\xi^2) - T(\xi^2)x^3 = \begin{pmatrix} 6 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ -37 \end{pmatrix}.$$

max $-11\sigma_1^1 + 13\sigma_1^2$ T.Q. $-2\sigma_1^1 + 3\sigma_1^2 \leq 0,$ $-1\sigma_1^1 + 2\sigma_1^2 \leq 0,$ $2\sigma_1^1 - 5\sigma_1^2 \leq 0,$ $1\sigma_1^1 - 6\sigma_1^2 \leq 0,$ $1\sigma_1^1 + 1\sigma_1^2 \leq 1$	maximum is at $\sigma_1^T = (\sigma_1^1, \sigma_1^2) = (0, 0)$
max $0\sigma_2^1 - 33\sigma_2^2$ T.Q. $-2\sigma_2^1 + 3\sigma_2^2 \leq 0,$ $-1\sigma_2^1 + 2\sigma_2^2 \leq 0,$ $2\sigma_2^1 - 5\sigma_2^2 \leq 0,$ $1\sigma_2^1 - 6\sigma_2^2 \leq 0,$ $1\sigma_2^1 + 1\sigma_2^2 \leq 1$	maximum is at $\sigma_2^T = (\sigma_2^1, \sigma_2^2) = (0, 0)$

The solution is feasible for both first and second scenario. To test the optimality of $x^3 = (6, 5)$, the dual (6) is solved for ξ^1 and ξ^2.

max $-13\pi_1^1 + 12\pi_1^2$ T.Q. $-2\pi_1^1 + 3\pi_1^2 \leq 1,$ $-1\pi_1^1 + 2\pi_1^2 \leq 0,$ $2\pi_1^1 - 5\pi_1^2 \leq 6,$ $1\pi_1^1 - 6\pi_1^2 \leq 2$	maximum is at $\pi_1^T = (\pi_1^1, \pi_1^2) = \left(-1, -\frac{1}{2}\right)$
max $0\pi_2^1 - 38\pi_2^2$ T.Q. $-2\pi_2^1 + 3\pi_2^2 \leq 5,$ $-1\pi_2^1 + 2\pi_2^2 \leq 3,$ $2\pi_2^1 - 5\pi_2^2 \leq 2,$ $1\pi_2^1 - 6\pi_2^2 \leq 1$	maximum is at $\pi_2^T = (\pi_2^1, \pi_2^2) = (0, -0.17)$

We calculate the value of the second stage problem associated with the two scenarios $Q(x^3, \xi^1)$ and $Q(x^3, \xi^2)$ respectively:

$$\mathbf{Q}(x^3, \xi^2) = \pi_1^T[h(\xi^1) - T(\xi^1)x^3] = \left(-1, -\frac{1}{2}\right)\begin{pmatrix} -13 \\ 12 \end{pmatrix} = 7,$$

$$\mathbf{Q}(x^3, \xi^1) = \pi_2^T[h(\xi^2) - T(\xi^2)x^3] = (0, -0.17)\begin{pmatrix} 0 \\ -38 \end{pmatrix} = 6.46,$$

$$\mathbf{Q}(x^3) = \frac{1}{2}\mathbf{Q}(x^3, \xi^1) + \frac{1}{2}\mathbf{Q}(x^3, \xi^1) = 6.73.$$

Then, $x^3 = (6, 5)$ is the optimal basic feasible solution on the current region. Let $\widetilde{Z}^3(x^3) = (-21, 23) + 6.73 = (-14.27, 29.73)$, $\widetilde{\Phi}(6, 5) = -2 + \theta = 4.73$, $\widetilde{\Phi}(6, 5) < \widetilde{\Phi}_{opt}$. We compute the weighted vector λ^3 of the $\widetilde{Z}^3$: $\lambda^3 = (0.87, 0.13)$.

- **Step 2.** We solve the generalized Tchebychev program $(P_\rho(\lambda^3))$, which is defined as follows:

$$(P_\rho(\lambda^3)) \begin{cases} \min & \alpha + 0.001(-3.7765 - 2x_1 + 2x_2) \\ \text{s.t.} & \alpha \geq 0.87(-16.765 + x_1 + 3x_2), \\ & \alpha \geq 0.13(13 - 3x_1 - x_2), \\ & x \in D. \end{cases} \qquad (23)$$

Then, $\bar{x}^3 = (4,6)$ is the optimal basic feasible solution of $(P_\rho(\lambda^3))$. To test the feasibility of the second-stage problems (15) and (16), we solve the program (7) with:

$$h(\xi^1) - T(\xi^1)\bar{x}^3 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} -13 \\ 7 \end{pmatrix},$$

$$h(\xi^2) - T(\xi^2)\bar{x}^3 = \begin{pmatrix} 6 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -35 \end{pmatrix}.$$

max	$-13\sigma_1^1 + 7\sigma_1^2$	
T.Q.	$-2\sigma_1^1 + 3\sigma_1^2 \leq 0,$	
	$-1\sigma_1^1 + 2\sigma_1^2 \leq 0,$	maximum is at
	$2\sigma_1^1 - 5\sigma_1^2 \leq 0,$	$\sigma_1^T = (\sigma_1^1, \sigma_1^2) = (0,0)$
	$1\sigma_1^1 - 6\sigma_1^2 \leq 0,$	
	$1\sigma_1^1 + 1\sigma_1^2 \leq 1$	

max	$2\sigma_2^1 - 35\sigma_2^2$	
T.Q.	$-2\sigma_2^1 + 3\sigma_2^2 \leq 0,$	
	$-1\sigma_2^1 + 2\sigma_2^2 \leq 0,$	maximum is at
	$2\sigma_2^1 - 5\sigma_2^2 \leq 0,$	$\sigma_2^T = (\sigma_2^1, \sigma_2^2) = (0,0)$
	$1\sigma_2^1 - 6\sigma_2^2 \leq 0,$	
	$1\sigma_2^1 + 1\sigma_2^2 \leq 1$	

The solution is feasible for both first and second scenario.
To test the optimality of $\bar{x}^3 = (4,6)$, the dual (6) is solved for ξ^1 and ξ^2.

max	$-13\pi_1^1 + 7\pi_1^2$	
T.Q.	$-2\pi_1^1 + 3\pi_1^2 \leq 1,$	maximum is at
	$-1\pi_1^1 + 2\pi_1^2 \leq 0,$	$\pi_1^T = (\pi_1^1, \pi_1^2) = \left(-1, -\frac{1}{2}\right)$
	$2\pi_1^1 - 5\pi_1^2 \leq 6,$	
	$1\pi_1^1 - 6\pi_1^2 \leq 2$	

max	$2\pi_2^1 - 35\pi_2^2$	
T.Q.	$-2\pi_2^1 + 3\pi_2^2 \leq 5,$	maximum is at
	$-1\pi_2^1 + 2\pi_2^2 \leq 3,$	$\pi_2^T = (\pi_2^1, \pi_2^2) = (0, -0.17)$
	$2\pi_2^1 - 5\pi_2^2 \leq 2,$	
	$1\pi_2^1 - 6\pi_2^2 \leq 1$	

We calculate the value of the second stage problem associated with the two scenarios $Q(\bar{x}^3, \xi^1)$ and $Q(\bar{x}^3, \xi^2)$ respectively:

$$\mathbf{Q}(\bar{x}^3, \xi^1) = \pi_1^T[h(\xi^1) - T(\xi^1)\bar{x}^3] = \left(-1, -\frac{1}{2}\right)\begin{pmatrix} -13 \\ 7 \end{pmatrix} = 9.5,$$

$$\mathbf{Q}(\bar{x}^3, \xi^1) = \pi_2^T[h(\xi^2) - T(\xi^2)\bar{x}^3] = (0, -0.17)\begin{pmatrix} 2 \\ -35 \end{pmatrix} = 5.95,$$

$$\mathbf{Q}(\bar{x}^3) = \frac{1}{2}\mathbf{Q}(\bar{x}^3, \xi^1) + \frac{1}{2}\mathbf{Q}(\bar{x}^3, \xi^1) = 7.725.$$

The solution $\widetilde{Z}^3(\bar{x}^3) = (-22, 18) + 5.88 = (-14.275, 25.725)$ is a nondominated point with minimal weighted Tchebychev distance, we obtain $\bar{x}^3 = (4,6)$ and $\widetilde{\Phi}(4,6) = 2 + 7.725 = 9.725$.

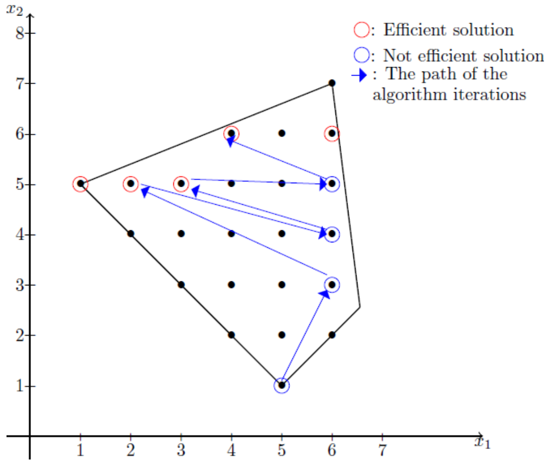

Fig. 3. The end iteration

- **Step 3.** Solve the equivalent efficient solutions program:

$$(T^3)\begin{cases} \min & \Phi(x) = -x_1 + x_2 + \theta \\ \text{s.t.} & x_1, x_2 \in D, \\ & -x_1 - 3x_2 + \theta = -14.275, \\ & 3x_1 + 1x_2 + \theta = 25.725. \end{cases} \tag{24}$$

An optimal solution is $x^{*3} = x^3 = (4, 6)$ with $\theta = 7.725$, $\widetilde{\Phi}(x^{*3}) = 9.725 > \widetilde{\Phi}_{opt} = 7.88$.
Terminated:= True

$x_{opt} = x^{*2} = (3, 5)$, $\widetilde{\Phi}_{opt} = 7.88$ is the optimal solution of the problem (P_E) (fig. 3).

The set of problems presented above have been solved by the MATLAB 7.0 environment. However, our algorithm optimizes the linear function $\Phi(x) = -x_1 + x_2$ without having to determine all these solutions but only $\mathbf{E}_s^3 = \{(2, 5), (3, 5), (4, 6)\}$.

Conclusion

In this work we have presented a new exact algorithm for optimizing over the integer efficient set of a stochastic multi-objective program based on the augmented weighted Tchebychev Program. The algorithm finds an integer optimal solution for problem (P_E) in a finite number of steps.

The proposed algorithm is formed on, the stochastic data are treated by recourse approach to obtain an equivalent deterministic program. We achieve this objective by combining two ideas: one consists of solving the augmented weighted Tchebychev program in the outcome space criteria to characterize nondominated criterion vector; then adding successive Gomory cuts, if necessary, we obtain an integer feasible solution and the feasibility cuts eliminate some parts of the first-stage decision set. And the second idea is to reduce progressively the admissible domain by adding more constraints eliminating all the points dominated by the current solution. A small number of iterations is necessary to obtain the optimal solution for (P_E).

In some applications the decision maker does not often have the possibility to use recours in the future, after the occurrence of a scenario (all K induced stresses are empty). In this case, we have to use another approach to the stochastic programming to convert the problem to a deterministic stochastic problem. Concerning the complexity, as we are obliged to transform the stochastic problem into deterministic one, problem (P_E) remains hard as was stated in deterministic case by Guyen (1992).

Acknowledgements. This work was supported in part by the Ministry of Higher Education and scientific research of Algeria. The authors would also like to thank Professor Farouk Yalaoui and Dr. Faicel Hnaien from LOSI University of Technology of Troyes, France, for their help and encouragement during the preparation of this paper.

References

[1] **Johan Ph.** Algorithms for the vector maximization problem. Mathematical Programming. 1972; 2(1):207–229.

[2] **Isermann H., Steuer R.E.** Computational experience concerning payoff tables and minimum criterion values over the efficient set. European Journal of Operational Research. 1988; 33(1):91–97.

[3] **Benson H.P.** Optimization over the efficient set. Journal of Mathematical Analysis and Applications. 1984; 98(2):562–580.

[4] **Ecker J.G., Song J.H.** Optimizing a linear function over an efficient set. Journal of Optimization Theory and Applications. 1994; 83(3):541–563.

[5] **Bolintineanu S.** Minimization of a quasi-concave function over an efficient set. Mathematical Programming. 1993; 61(1–3):89–110.

[6] **Sayin S.** Optimizing over the efficient set using a top-down search of faces. Operations Research. 2000; 48(1):65–72.

[7] **Abbas M., Chaabane D.** Optimizing a linear function over an integer efficient set. European Journal of Operational Research. 2006; 174(2):1140–1161.

[8] **Jorge J.M.** An algorithm for optimizing a linear function over an integer efficient set. European Journal of Operational Research. 2009; 195(1):98–103.

[9] **Chaabane D., Brahmi B., Ramdani Z.** The augmented weighted tchebychev norm for optimizing a linear function over an integer efficient set of a multicriteria linear program. International Transactions in Operational Research. 2012; 19(4):531–545.

[10] **Teghem J.** Strange-momix: An interactive method for mixed integer linear programming. Stochastic VerKall – Mayersus fuzzy apprroaches to multiobjective mathematical programming under uncertainty. Dordrecht: Kluwer; 1990: 101–115.

[11] **Abbas M., Bellahcene F.** Cutting plane method for multiple objective stochastic integer linear programming. European Journal of Operational Research. 2006: 168(3):967–984.

[12] **Abbas M., Moulai M.** An algorithm for mixed integer linear fractional programming problem. Jorbel. 1999; (39):21–30.

[13] **Van Slyke R.M., Wets R.** L-shaped linear programs with applications to optimal control and stochastic programming. SIAM Journal on Applied Mathematics. 1969; 17(4):638–663.

[14] **Amrouche S., Moulai M.** Multi-objective stochastic integer linear programming with fixed recourse. International Journal of Multicriteria Decision Making. 2012; 2(4):355–378.

[15] **Benders J.F.** Partitioning procedures for solving mixed-variables programming problems. Numerische Mathematik. 1962; 4(1):238–252.

[16] **Steuer R.E.** Multiple criteria optimization: Theory, computation, and applications. Wiley; 1986: 546.

[17] **Kall P., Mayer J.** Stochastic linear programming. Springer, 1976; (21):92.

[18] **Kall P., Wallace S.W.** Stochastic programming. Journal of the Operational Research Society. 1994; 46(9). DOI:10.2307/2584504.

[19] **V Joseph Bowman Jr.** On the relationship of the Tchebycheff norm and the efficient frontier of multiple-criteria objectives. In Multiple criteria decision making. Springer; 1976: 76–86.

[20] **Steuer R.E., Choo E.-U.** An interactive weighted tchebycheff procedure for multiple objective programming. Mathematical Programming. 1983; 26(3):326–344.

[21] **Sylva J., Crema A.** A method for finding the set of non-dominated vectors for multiple objective integer linear programs. European Journal of Operational Research. 2004; 158(1):46–55.

Вычислительные технологии, 2020, том 25, № 6, с. 86–106. © ФИЦ ИВТ, 2020 ISSN 1560-7534
Computational Technologies, 2020, vol. 25, no. 6, pp. 86–106. © FRC ICT, 2020 eISSN 2313-691X

ВЫЧИСЛИТЕЛЬНЫЕ ТЕХНОЛОГИИ

DOI:10.25743/ICT.2021.26.3.006

Стохастическая оптимизация по фронту Парето с помощью расширенной взвешенной программы Чебышева

Л. Юнси-Аббаси*, М. Мула

Университет науки и технологии Уари Бумедьен, лаборатория LaROMaD, факультет математики, 16111, Баб-Эззуар, Эль-Алия, Алжир

* Контактный автор: Лейла Юнси-Аббаси, e-mail: abbaci.leila@yahoo.fr

Поступила 14 декабря 2020 г., доработана 21 апреля 2021 г., принята в печать 28 апреля 2021 г.

Аннотация

В этой статье мы предлагаем новый алгоритм для решения многоцелевых задач стохастического целочисленного линейного программирования (MOSILP). Мы оптимизируем данную стохастическую линейную функцию ϕ по полному набору эффективных решений MOSILP, которые были преобразованы в эквивалентную детерминированную задачу с использованием неопределенных предположений, вводимых лицом, принимающим решения. Для этой цели мы применяем двухэтапный рекурсивный подход, при котором расширенная взвешенная программа Чебышева постепенно оптимизируется для создания эффективного решения, тем самым улучшая значение вспомогательной функции ϕ. Предлагаемый здесь подход определяет и решает последовательность целочисленных линейных программ с нарастающими ограничениями, так что на каждом этапе алгоритма генерируется новое эффективное решение. Для иллюстрации представлен числовой пример.

Ключевые слова: многоцелевая задача, целочисленное программирование, стохастическое линейное программирование, норма Чебышева.

Цитирование: Younsi-Abbaci L., Moulaï M. Stochastic optimization over the Pareto front by the augmented weighted Tchebychev program. Computational Technologies. 2021; 26(3):86–106. DOI:10.25743/ICT.2021.26.3.006.

Благодарности. Исследование выполнено при частичной поддержке Министерства высшего образования и научных исследований Алжира. Авторы также выражают благодарность профессору Фаруку Ялауи и доктору Файселу Хнайену из Технологического университета LOSI (Труа, Франция) за их помощь и поддержку в подготовке данной статьи.