ON THE GENERALIZED HEAT KERNEL*

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В данной работе мы исследуем уравнение

$$\frac{\partial}{\partial t} u(x,t) = -c^2(-\Delta)^k u(x,t)$$

с начальными условиями

$$u(x,0) = f(x),$$

где $x \in \mathbb{R}^n$, $\mathbb{R}^n$ — $n$-мерное евклидово пространство. Оператор $\Delta^k$ называется оператором Лапласа, итерированным $k$ раз, и определяется как

$$\Delta^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k,$$

где $n$ — размерность евклидова пространства $\mathbb{R}^n$; $u(x,t)$ — неизвестная функция от $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times [0, \infty)$; $f(x)$ — заданная обобщенная функция; $k$ — неотрицательное целое число; $c$ — положительная постоянная.

Решение такого уравнения, называемое обобщенным ядром уравнения теплопроводности, имеет интересные свойства и связано с решением уравнения теплопроводности.

Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Delta u(x,t)$$

(0.1)

with the initial condition

$$u(x,0) = f(x),$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times [0, \infty)$, we obtain the solution

$$u(x,t) = \frac{1}{(4c^2 \pi t)^{n/2}} \int_{\mathbb{R}^n} \exp \left[ -\frac{|x-y|^2}{4c^2 t} \right] f(y) dy.$$
Alternatively, this solution can be represented in the convolution form

$$u(x, t) = E(x, t) * f(x),$$

(0.2)

where

$$E(x, t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp \left[ -\frac{|x|^2}{4c^2t} \right].$$

(0.3)

The function (0.3) called the heat kernel, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$, see [1, p. 208, 209].

Moreover, we obtain $E(x, t) \to \delta$ as $t \to 0$, where $\delta$ is the Dirac-delta function. We can extend (0.1) to the equation

$$\frac{\partial}{\partial t} u(x, t) = -c^2 \Delta^2 u(x, t)$$

(0.4)

with the initial condition

$$u(x, 0) = f(x),$$

where $\Delta^2 = \Delta \Delta$ is the biharmonic operator, that is

$$\Delta^2 = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^2.$$

Using the $n$-dimensional Fourier transform we can find the following solution of (0.4)

$$u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-c^2|\xi|^4t + i(\xi, x-y)} f(y) dy d\xi.$$ 

(0.5)

Using (0.5) $u(x, t)$ can be rewritten in the convolution form

$$u(x, t) = E(x, t) * f(x),$$

where

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-c^2|\xi|^4t + i(\xi, x)} d\xi,$$

(0.6)

$|\xi|^4 = (\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^2$ and $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$. The function $E(x, t)$ in (0.6) is the kernel of (0.4), $E(x, t) \to \delta$ as $t \to 0$ since

$$\lim_{t \to 0} E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{(\xi, x)i}, d\xi = \delta,$$

see [3, p. 396, Eq. (10.2.19(b))].

Now, the purpose of this work is to study the equation

$$\frac{\partial}{\partial t} u(x, t) = -c^2(-\Delta)^2 u(x, t)$$

(0.7)

with the initial condition

$$u(x, 0) = f(x), \text{ for } x \in \mathbb{R}^n,$$
where the operator $\triangle^k$ denotes the Laplace operator iterated $k$-times. This operator is defined as follows

$$\triangle^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k,$$

(0.8)

where $n$ is the dimension of Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown function, $(x,t) = (x_1,x_2,\ldots,x_n,t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is the given generalized function, $k$ is a nonnegative integer and $c$ is a positive constant.

We obtain $u(x,t) = E(x,t) * f(x)$ as a solution of (0.7), where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[ -c^2 \left( \sum_{i=1}^{p} \xi_i^2 \right)^k t + i(\xi,x) \right] d\xi.$$

(0.9)

All properties of $E(x,t)$ in (0.9) will be studied in details.

Now, if we set $k = 1$ in (0.9) then (0.9) reduces to (0.3), which is the kernel of (0.1). Also, if we set $k = 2$ in (0.9), then (0.9) reduces to (0.6), which is the kernel of (0.4).

1. Preliminaries

**Definition 1.1.** Let $f(x) \in L_1(\mathbb{R}^n)$ be the space of integrable functions in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x) dx,$$

(1.1)

where $\xi = (\xi_1,\xi_2,\ldots,\xi_n)$, $x = (x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$, $(\xi,x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the usual inner product in $\mathbb{R}^n$, $dx = dx_1 dx_2 \ldots dx_n$.

The inverse Fourier transform is given by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \hat{f}(\xi) d\xi.$$

(1.2)

**Lemma 1.1.** Given the function

$$f(x) = \exp \left[ - \left( \sum_{i=1}^{n} x_i^2 \right)^k \right],$$

where $(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$. Then

$$\left| \int_{\mathbb{R}^n} f(x) dx \right| \leq \frac{\pi^{\frac{n}{2}} \Gamma \left( \frac{n}{2k} \right)}{k \Gamma \left( \frac{n}{2} \right)},$$

(1.3)

where $\Gamma$ denotes the Gamma function. Therefore, $\int_{\mathbb{R}^n} f(x) dx$ is bounded.
Proof. We have
\[ \int_{\mathbb{R}^n} f(x) \, dx = \int_{\mathbb{R}^n} \exp \left[ - \left( \sum_{i=1}^{p} x_i^2 \right)^k \right] \, dx. \]

Let us transform to bipolar coordinates
\[ x_1 = r\omega_1, x_2 = r\omega_2, \ldots, x_n = r\omega_n, \]
where \( \sum_{i=1}^{n} \omega_i^2 = 1. \)

Thus
\[ \int_{\mathbb{R}^n} f(x) \, dx = \int_{\mathbb{R}^n} e^{-r^{2k} r^{-n-1}} \, dr \, d\Omega_n, \]
where
\[ dx = r^{n-1} \, dr \, d\Omega_n, \quad (1.4) \]

\( d\Omega_n \) is the element of surface area on the unit sphere in \( \mathbb{R}^n. \) By direct computation we obtain
\[ \int_{\mathbb{R}^n} f(x) \, dx = \Omega_n \int_{0}^{\infty} e^{-r^{2k} r^{-n-1}} \, dr, \quad (1.5) \]

where \( \Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \)

When \( u = r^{2k}, \) we then obtain
\[ \int_{\mathbb{R}^n} f(x) \, dx \leq \frac{\Omega_n}{2k} \int_{0}^{\infty} e^{-u \frac{2k}{n}} \, du = \frac{\Omega_n}{2k} \Gamma \left( \frac{n}{2k} \right) = \frac{\pi^{\frac{n}{2}}}{k} \frac{\Gamma \left( \frac{n}{2k} \right)}{\Gamma \left( \frac{n}{2} \right)}. \quad (1.6) \]

Therefore, \( \int_{\mathbb{R}^n} f(x) \, dx \) is bounded. \( \square \)

Lemma 1.2. For all \( t > 0 \) and all \( x \in \mathbb{R} \) we have
\[ \int_{-\infty}^{\infty} \exp \left( -c^2 \xi^2 t \right) \, d\xi = \sqrt{\frac{\pi}{c^2 t}} \quad (1.7) \]
and
\[ \int_{-\infty}^{\infty} \exp \left[ -c^2 \xi^2 t + i\xi x \right] \, d\xi = \sqrt{\frac{\pi}{c^2 t}} \exp \left( -\frac{x^2}{4c^2 t} \right), \quad (1.8) \]

where \( c \) is a positive constant.

Proof. See [2, p. 117, 118]. \( \square \)
2. Main Results

Theorem 2.1. Given the equation

\[ \frac{\partial}{\partial t} u(x, t) = -c^2 (-\Delta)^k u(x, t) \]  

(2.1)

with the initial condition

\[ u(x, 0) = f(x), \]  

(2.2)

where \( \Delta^k \) is the Laplace operator iterated \( k \)-times defined by

\[ \Delta^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k, \]

where \( n \) is the dimension of Euclidean space \( \mathbb{R}^n \), \( k \) is a nonnegative integer, \( u(x, t) \) is an unknown function, \( (x, t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty) \), \( f(x) \) is the given generalized function, and \( c \) is a positive constant. Then we obtain that

\[ u(x, t) = E(x, t) * f(x) \]  

(2.3)

is a solution of (2.1), which satisfies (2.2) where \( E(x, t) \) is the kernel of (2.1) defined by

\[ E(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp \left[ -c^2 \left( \sum_{i=1}^{n} \xi_i^2 \right)^k t + i(x, \xi) \right] d\xi. \]  

(2.4)

**Proof.** Applying the Fourier transform (1.1) to both sides of (2.1), we obtain

\[ \frac{\partial}{\partial t} \hat{u}(\xi, t) = -c^2 \left( \sum_{i=1}^{n} \xi_i^2 \right)^k \hat{u}(\xi, t). \]

Thus,

\[ \hat{u}(\xi, t) = K(\xi) \exp \left[ -c^2 \left( \sum_{i=1}^{n} \xi_i^2 \right)^k t \right], \]  

(2.5)

where \( K(\xi) \) is a constant and \( \hat{u}(\xi, 0) = K(\xi) \).

\( \hat{u}(\xi, t) \) in (2.5) is bounded and from (2.2) we have

\[ K(\xi) = \hat{u}(\xi, 0) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(x, \xi)} f(x) \, dx \]  

(2.6)

and using the inversion in (1.2) we obtain from (2.5) and (2.6)

\[ u(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(x, \xi)} \hat{u}(\xi, t) \, d\xi = \]

\[ = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x, \xi)} e^{-i(y, \xi)} f(y) \exp \left[-c^2 \left( \sum_{i=1}^{n} \xi_i^2 \right)^k t \right] \, dy \, d\xi. \]
Therefore,

\[ u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\xi, x - y)} \exp \left[ -c^2 \left( \sum_{i=1}^{n} \xi_i^2 \right) t \right] f(y) \, dy \, d\xi \]  

(2.7)

or

\[ u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp \left[ -c^2 \left( \sum_{i=1}^{n} \xi_i^2 \right)^k t + i(\xi, x - y) \right] f(y) \, dy \, d\xi. \]  

(2.8)

Set

\[ E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[ -c^2 \left( \sum_{i=1}^{n} \xi_i^2 \right) t + i(\xi, x) \right] d\xi. \]  

(2.9)

Thus, (2.8) can be rewritten in the convolution form

\[ u(x, t) = E(x, t) \ast f(x), \]  

(2.10)

where \( u(x, t) \) in (2.8) is a solution of (2.1) and \( E(x, t) \) is defined by (2.9). It is clear that the kernel \( E(x, t) \) exists.

Moreover, since \( E(x, t) \) exists, then

\[ \lim_{t \to 0} E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\xi, x)} \, d\xi = \delta(x), \text{ for } x \in \mathbb{R}^n. \]  

(2.11)

See [3, p. 396, Eq. (10.2.19(b))].

From (2.11) we obtain

\[ u(x, 0) = \lim_{t \to 0} u(x, t) = \lim_{t \to 0} (E(x, t) \ast f(x)) = \delta \ast f(x) = f(x). \]

Thus, \( u(x, t) \) in (2.3) satisfies (2.2).

In particular, if we set \( k = 1 \) in (2.9), then we obtain

\[ E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[ -c^2 \left( \sum_{j=1}^{n} \xi_j^2 \right) t + i(\xi, x) \right] d\xi = \]

\[ = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[ -c^2 \sum_{j=1}^{n} \xi_j^2 t + i \sum_{j=1}^{n} \xi_j x_j \right] d\xi = \]

\[ = \frac{1}{(2\pi)^n} \prod_{j=1}^{n} \int_{-\infty}^{\infty} \exp \left[ -c^2 \xi_j^2 t + i \xi_j x_j \right] d\xi_j = \]

\[ = \frac{1}{(2\pi)^n} \prod_{j=1}^{n} \sqrt{\frac{\pi}{c^2 t}} \exp \left( -\frac{x_j^2}{4c^2 t} \right). \]

from (1.8). Thus,

\[ E(x, t) = \frac{1}{(4c^2 \pi t)^{n/2}} \exp \left( -\frac{|x|^2}{4c^2 t} \right), \]
since

\[(\frac{\pi}{c^2 t})^{\frac{n}{2}} \exp \left(-\frac{|x|^2}{4c^2 t}\right) = \prod_{j=1}^{n} \sqrt{\frac{\pi}{c^2 t}} \exp \left(-\frac{x_j^2}{4c^2 t}\right)\]

and \(|x|^2 = \sum_{i=1}^{n} x_i^2\).

Therefore, if we set \(k = 1\) in (2.1) and (2.9), then (2.1) and (2.9) will be reduced to (0.1) and (0.3), respectively. If we set \(k = 2\) in (2.9), then we obtain

\[E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[-c^2 \left(\sum_{i=1}^{n} \xi_i^2 \right)^{\frac{k}{2}} t + i(\xi, x)\right] d\xi =\]

\[= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-c^2 |\xi|^4 t + i(\xi, x)} d\xi,\]

where \(|\xi|^4 = (\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^2\).

Therefore, if we set \(k = 2\) in (2.1) and (2.9), then (2.1) and (2.9) will be reduced to (0.4) and (0.6), respectively. \(\square\)

**Theorem 2.2.** The kernel \(E(x, t)\) defined by (2.9) has the following properties:

1) \(E(x, t) \in C^\infty\), where \(C^\infty\) is the space of continuous infinitely differentiable functions, \(x \in \mathbb{R}^n, t > 0;\)

2) \(\left(\frac{\partial}{\partial t} + c^2 (-\Delta)^k\right)E(x, t) = 0\) for \(t > 0;\)

3) \(E(x, t) > 0\) for \(t > 0;\)

4) \(|E(x, t)| \leq \frac{1}{2^n \pi^{n/2} k (c^2 t)^{n/2k}} \frac{\Gamma \left(\frac{n}{2k}\right)}{\Gamma \left(\frac{n}{2}\right)}, \text{ for } t > 0,\)

where \(\Gamma\) denotes the Gamma function. Thus \(E(x, t)\) is bounded for any fixed \(t;\)

5) \(\lim_{t \to 0} E(x, t) = \delta.\)

**Proof.**

1. This property follows from (2.9), since

\[\frac{\partial^n}{\partial x^n} E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\partial^n}{\partial x^n} \exp \left[-c^2 \left(\sum_{i=1}^{n} \xi_i^2 \right)^{\frac{k}{2}} t + i(\xi, x)\right] d\xi.\]

Thus, \(E(x, t) \in C^\infty\) for \(x \in \mathbb{R}^n, t > 0.\)

2. By direct computation we obtain

\[\left(\frac{\partial}{\partial t} + c^2 (-\Delta)^k\right)E(x, t) = 0\]

for \(t > 0,\) where \(E(x, t)\) is defined by (2.9).

3. \(E(x, t) > 0\) for \(t > 0\) is obvious from (2.9).
4. From (2.9) we have

\[ E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[ -c^2 \left( \sum_{i=1}^{n} \xi_i^2 \right)^k t + i(\xi, x) \right] d\xi. \]

Therefore,

\[ |E(x, t)| \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[ -c^2 t \left( \sum_{i=1}^{n} x_i^2 \right)^k \right] dy. \]

Using the same procedure as in Lemma 1.1, we obtain

\[ |E(x, t)| \leq \frac{1}{2^n \pi^{n/2} k^{n/2} t^{n/2}} \Gamma \left( \frac{n}{2k} \right). \]

Thus, \( E(x, t) \) is bounded for any fixed \( t \).

5. This property is obvious from (2.11). \( \square \)

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References


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