ANALYSIS OF STRESS STATE WITH THE FORCE LINES METHOD

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A numerical method to build up lines of force field in deformed solids is proposed. Distribution of the force lines is determined on the basis of known stress state calculated by any numerical method or analytical solution. Two possible applications of the method are considered: analysis of stress state and the optimal placement of reinforced elements in constructions for obtaining desirable mechanical properties.

The method of force lines

To built up the field of force lines in deformed solid bodies the possibility of calculating the value of stress components at any point is necessary. The stress field can be obtained by any numerical method (FEM, BIM etc.) or analytical solution. The procedure for determining the force lines consists of two stages.

The fist stage implies the calculation of the trajectory S on which the intersection points of the lines are found out. This can be done in a number of ways. For instance, it can be defined as the lines perpendicular to the maximum principal stresses acting at each point of a body (σ_1), or the lines perpendicular to the maximal principal stresses in terms of absolute value $\sigma_n = \max(|\sigma_1|, |\sigma_2|)$. If the principal directions are taken as the spatial axis the trajectories S coincide with the x or y axis.

If the location of the first point A_{ij} on S_j is known this trajectory may be determined by the equations:

$$dx = ds \cdot \cos(\alpha(x,y) \pm \pi/2),$$
$$dy = ds \cdot \sin(\alpha(x,y) \pm \pi/2), \tag{1}$$

where dx and dy are the increments of the point coordinates along the S_j , $\alpha(x,y)=\alpha(\sigma_x, \sigma_y, \sigma_{xy})$ is the angle defining the principal directions at the point [1]:

$$\frac{2 \cdot \tau_{xy}}{\sigma_x - \sigma_y} = \tan 2\alpha, \tag{2}$$

The choice of the position of the initial point A_{ij} depends on the character of the stress state, on the part geometry and on the given accuracy. The iterative procedure can be carried out until the location of this point does not change the positions the trajectory S_j. For confidence it is better to choose this point on the part edge where the stress state is uniform.

The second stage involves calculating the force lines L_i intersecting the trajectory S and defined by the point A_{ij}. The successive point A_{i+1j} is found by integrating normal stresses along the trajectory S_j. In the general case a location of point A_{i+1j} with respect to A_{ij} for a plate with unit thickness is determined by the condition:

$$\int_{A_{ij}}^{A_{ij+1}} \sigma_n ds = R = const \tag{3}$$

where ds is the element of the path S; σ_n is the normal stress acting on ds ; R is the force magnitude – the "cost" of a force line. Thus the points A_{i+kj} divide the trajectory S_j on parts where the total force of magnitude R acts. The value of R can be constant or changes for different S depending on a type of analysis. Whereas the fist case implies that the number of the force lines can change at different sections of a body (for example, the density of the force lines is higher in a zone of stress concentration than in one with nominal stresses or there is no line if the sum force in a section is less than R), the latest assumes the same number of them.

By successively repeating the stages of finding the points A_{ij} on different trajectories S and linking corresponding points the whole picture of force lines distribution in deformed body is created (Fig.1).

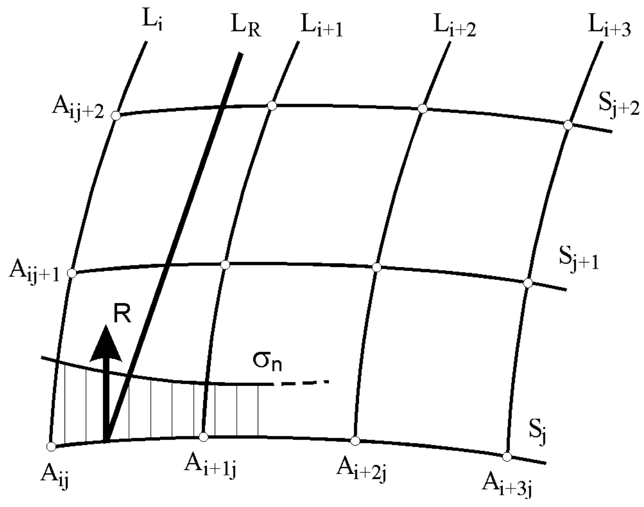

Fig.1. The scheme of building of the force lines L_i and L_R

Since there are no analytical solutions of the stress components in many cases of loads in numerical procedure Δx, Δy, Δs are used instead of dx, dy, ds in equation (1). Their values determine the accuracy of force lines building. So the lines L_i divide the trajectory S_i into sections the constant sum force equals to R, it is possible to determine the location of the point A_{Ri} - the place where a moment from normal stress acting on S_j between A_{i+1j} is zero:

$$\int_{A_{ij}}^{A_R} \sigma_n \cdot s ds = \int_{A_R}^{A_{ij+1}} \sigma_n \cdot s ds \tag{4}$$

Through the points A_R one can draw the line L_R that shows the locations of the acting of the force with magnitude R.

Consider as an example the force lines building for the case of pure bending of a beam with rectangular cross section. For that case there is only one stress component parallel to the axis x with linear stress distribution, proportional to the distance from the neutral line. The trajectories S for the case are laying on z-axis and conditions (3,4) are respectively:

$$\int_{A_i}^{A_{i+1}} \sigma \ dz = R = \frac{M \cdot h^2}{2 \cdot I \cdot n},$$

$$\int_{A_i}^{A_{Ri}} \sigma \cdot z dz = \int_{A_{Ri}}^{A_{i+1}} \sigma \cdot z dz \quad \text{or} \quad A_{Ri} = \sqrt[3]{\frac{A_{i+1}^3 + A_i^3}{2}} \tag{5}$$

where M – the bending moment; I – secondary moment of inertia; h – half of the beam height; n – number of force lines. The L_i and L_R lines for the case with $n=4$ are shown on Fig.2.

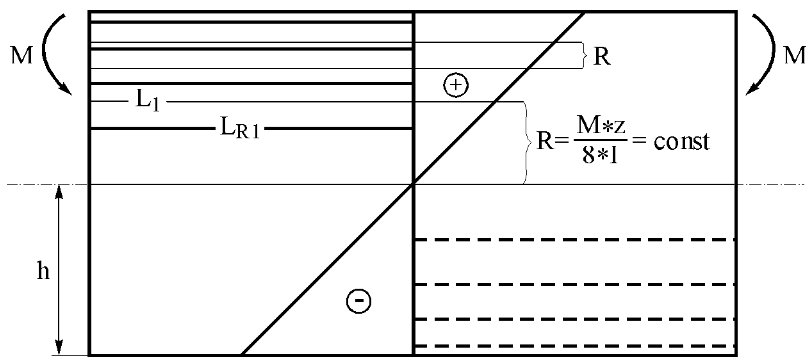

Fig. 2. The distribution of the force lines in the beam under pure bending

Force lines in anisotropic plates with crack

Force lines analysis provides good opportunities to give insight into effect of material anisotropy on the stress state near openings and cracks. Calculations based on the complete solution for a crack in orthotropic plate [2] were carried out for two materials having different the longitudinal to transverse modulus relationship. Stress distributions of the σ_y stress component at a line of crack continuation are identical for both materials. Whereas the σ_x components essentially different and if the crack lies along direction with larger rigidity, σ_x can exceed σ_y. For the unidirectional composite with a crack passing across the fiber direction the force lines tend to stretch along the fibers. Thus the transition area from the uniform distribution of force lines to that one near crack is wider than for a material having a crack along fiber direction (Fig. 3). In the second case (Fig. 3b) the force lines near the crack tip exhibit higher density. For the crack perpendicular to the fibers a force lines distribution is similar to their distribution in plate with a hole in isotropic material [3].

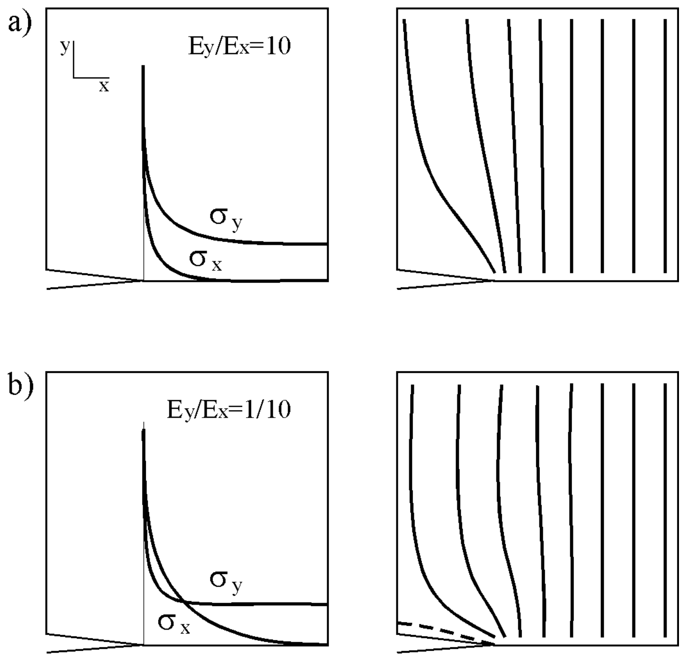

Fig. 3. Force lines field in othotropic plates under tension calculated from the main stress maximal in absolute value.

The optimal placement of reinforced elements using the method of force lines

Materials generally used in Integrated Processing [4] as reinforcing elements (tows) have much higher mechanical properties than the basis material. For example in technology of the tow placement with over injection, commingled 69% carbon/PA12 UD with elastic modulus 110 GPa in fiber direction and PA12 as a basis material with modulus 1.1 GPa are used. Because of the difference in elastic modulus most loading acting on construction will be taking by the tows. The tows placed in construction under loading have a similar character as force lines – they are taking the result force acting in the central section of a construction.

The force lines method presented above is a numerical tool for analysis of stress state of construction under external loading. It seems possible to apply this method for determination of the optimal placement of reinforced elements assuming the tows as the force lines in the deformed construction. However there are some essential differences between the force lines and reinforced elements. The most important one is that whereas the force lines are dimensionless (no thickness and width) the tows have a section of a certain size. For that case a setting of correctness (accordance) between the L_R and the tows can be done in several ways. One of them is to consider the tow as the force line with width a and unit force $R'=R/a$. This also gives the possibility to calculate the optimal placement of tows with variable section.

The other important difference is that in construction reinforced by the tows the basis material bears a certain part of applied loading. The force lines distribution is determined for a deformed part without reinforced elements. Therefore after placement of tows a new system (new construction) is created with a new stress state. It is possible to estimate where and how much a placement of tows will change the force lines distribution. Taking into account very high relation of fiber and matrix moduli those changes will be minimal in place with uniform stress distribution and maximal in place where is a large gradient of stress concentration or complex stress state.

In general the procedure of determination of the optimal tow placement is iterative. Successive recalculations of the force lines location for new created systems must be done. The reason to stop this iterative process can be different and depends on concrete problem of tow placement. For instance, when the ultimate possible tow configurations have been reached, or when the new distribution of force lines does not differ very much (less than given value) from the old one.

It is important to mention that the practical realization of the tow placement in Integrated Processing creates essential limitations on the choice of reinforcement criteria (what king of advantage or improvement is intended to obtain by the tow placement) and therefor on the method and accuracy of the force lines building. Those are the placement speed; the ultimate geometrical values of the tow configuration (for instance, the critical value of bending tow radius); the precision of equipment to put the tows at given path and so on. On the recent stage of Integration Processing development it is difficult to make the tow placement discontinuously, it also increases the cost of processing. Therefore it seems reasonable to use the force lines method with variable value of the magnitude *R* – constant number of tows in each section of a part even though it leads to placement of the town in places with relatively low stress level. Using the tows with variable section can solve that problem.

Consider practical application of the force lines method for a part – the "hook" – under given loading conditions. The stress field in the part has been calculated with the method of finite elements assuming the plane strain conditions. Figure 4 shows load conditions and the positions of the force lines in the hook calculated with different methods of the force lines building. The y-axis as the principal direction has been taken so the line S coincide with x-axis. The direction of the force lines obtained by using the method with constant number of force lines $L_R=4$ (it implies that the magnitude *R* is various on different trajectories *S*) for analysis of tensile stresses is presented on Fig.4, a.

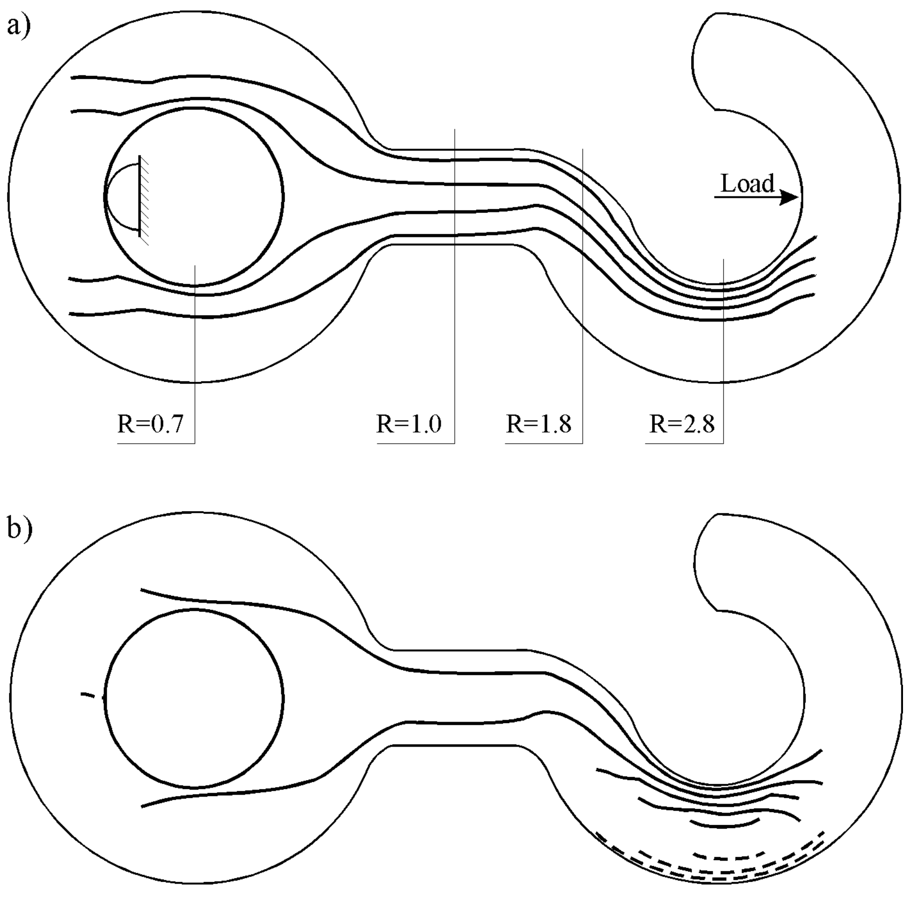

Fig. 4. Trajectories of the force lines in the hook. The analysis with constant number of the force lines (a) and constant magnitude *R* (b). The force lines of the compressive stresses is shown by dash line.

The unit of the force line was determined by a calculation of total force in each section of the hook. The relative value of force line $R=1$ has been taken in section with more or less uniform stress state. The drawing of the lines was stopped if the force line value in a section was less then a half of a unit. The problem of limiting the displacements in loading direction can be considered as a practical example of application of such kind analysis. An interesting idea of application of the force lines method with variable value of *R* can be also consider. It is known that void content in the tow strongly depends on the speed of placement. The void content itself influences on stiffness and especially fracture characteristics of the tows. The variable *R*-value for the force line means that the tow placed on its trajectory will bear a loading of different intensity in different places. Therefore the speed of tow placement can be faster in place where the magnitude of *R* is less. It brings to idea to create constructions with distribution of mechanical properties and uniform distribution of fracture properties.

The case when the value of a force line is constant in each section and both tensile and compressive stresses were examined is presented on Fig.4, b. The number of the force lines increases in areas with stress concentration. This analysis can be applied to the problem of achievement of certain level of ultimate loading when a failure depends on both compressive and tensile stresses.

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